ECCS Warm-up

Alex Arenas
Universitat Rovira i Virgili (Tarragona, Spain)

Mesoscopic description of complex networks: theory and applications

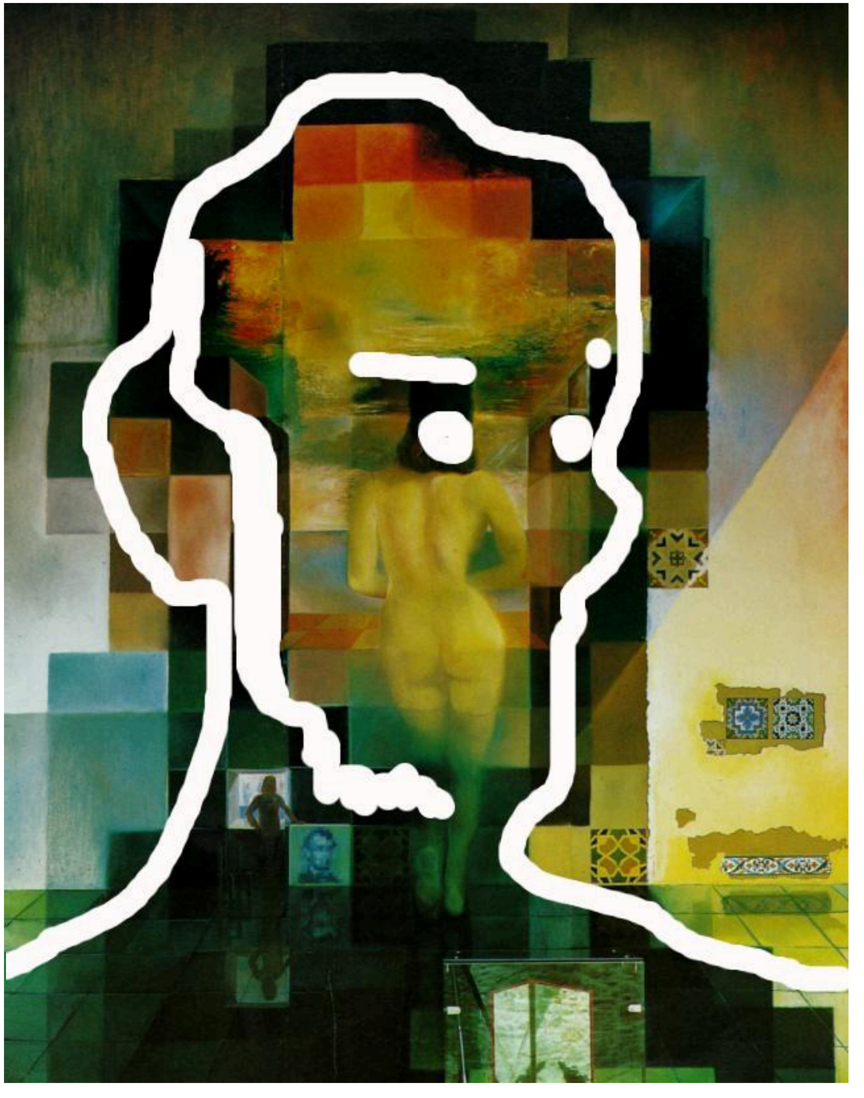
Summary

- Introduction
- Community detection problem
- Dynamic mesoscales
- Static mesoscales
- Applications

What it means “mesoscopic”?

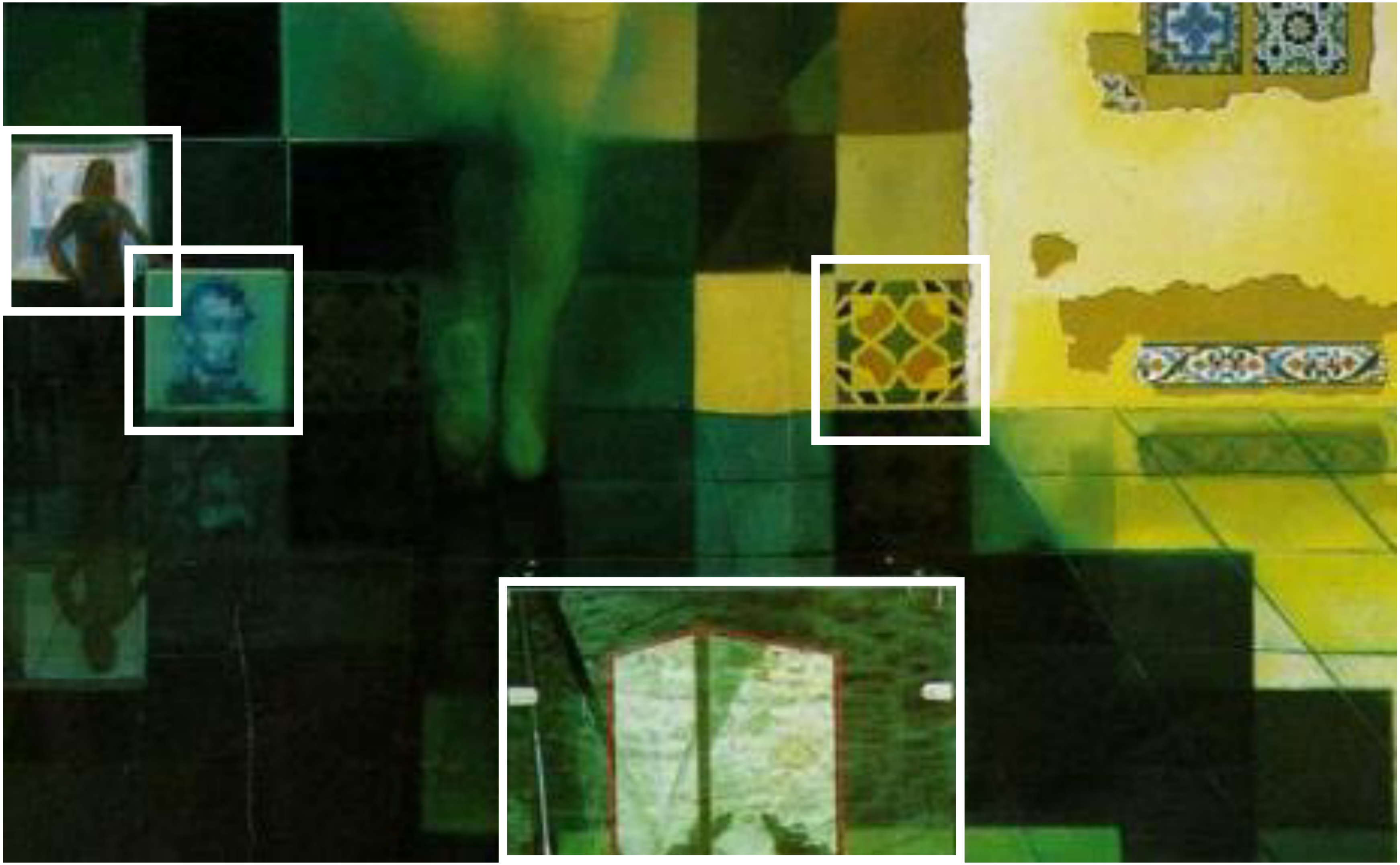
- Difficult to assess an accurate definition:
  - Statics’ perspective: consideration of length scales ranging between max and min levels of experimental resolution
  - Dynamics’ perspective: consideration of dynamic phenomena that appear at time or length scales that range between the max and min levels of description for the evolution

Static example: Dali -Lincoln portrait-



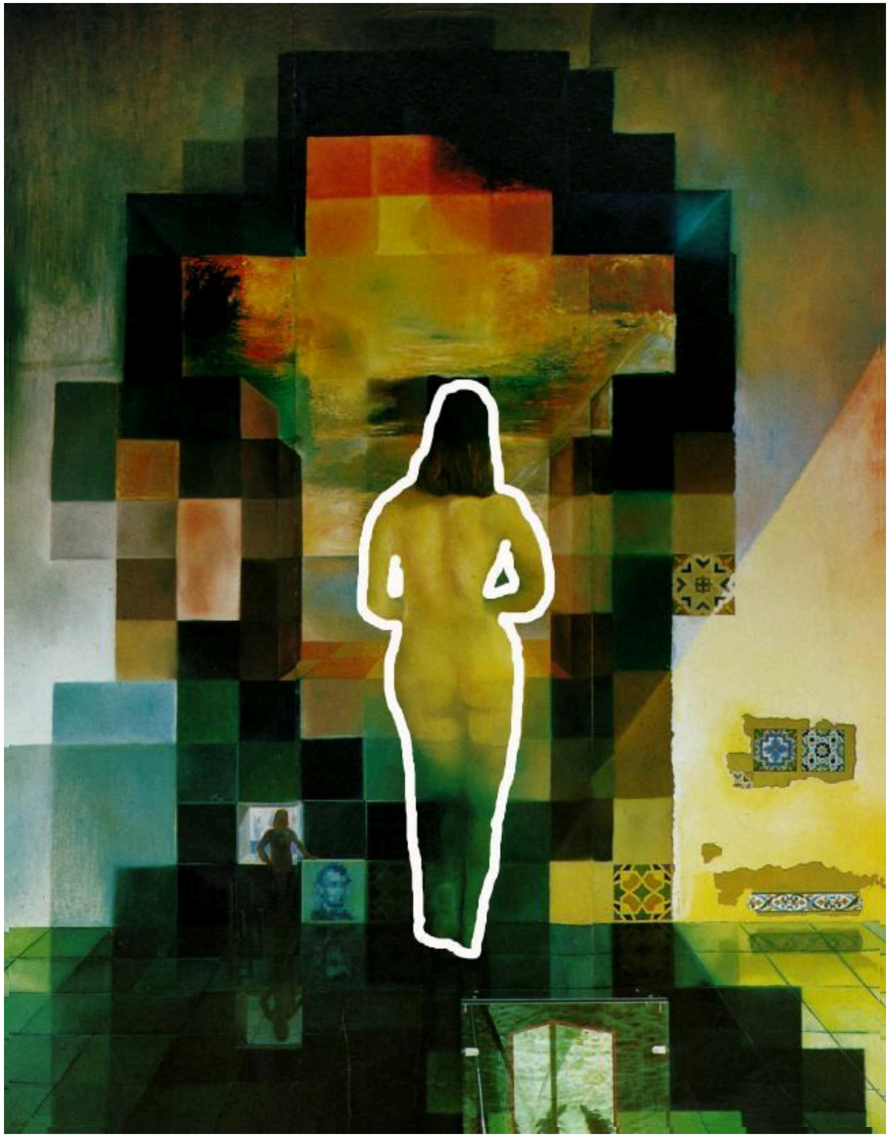
Macroscale

Static example: Dali -Lincoln portrait-



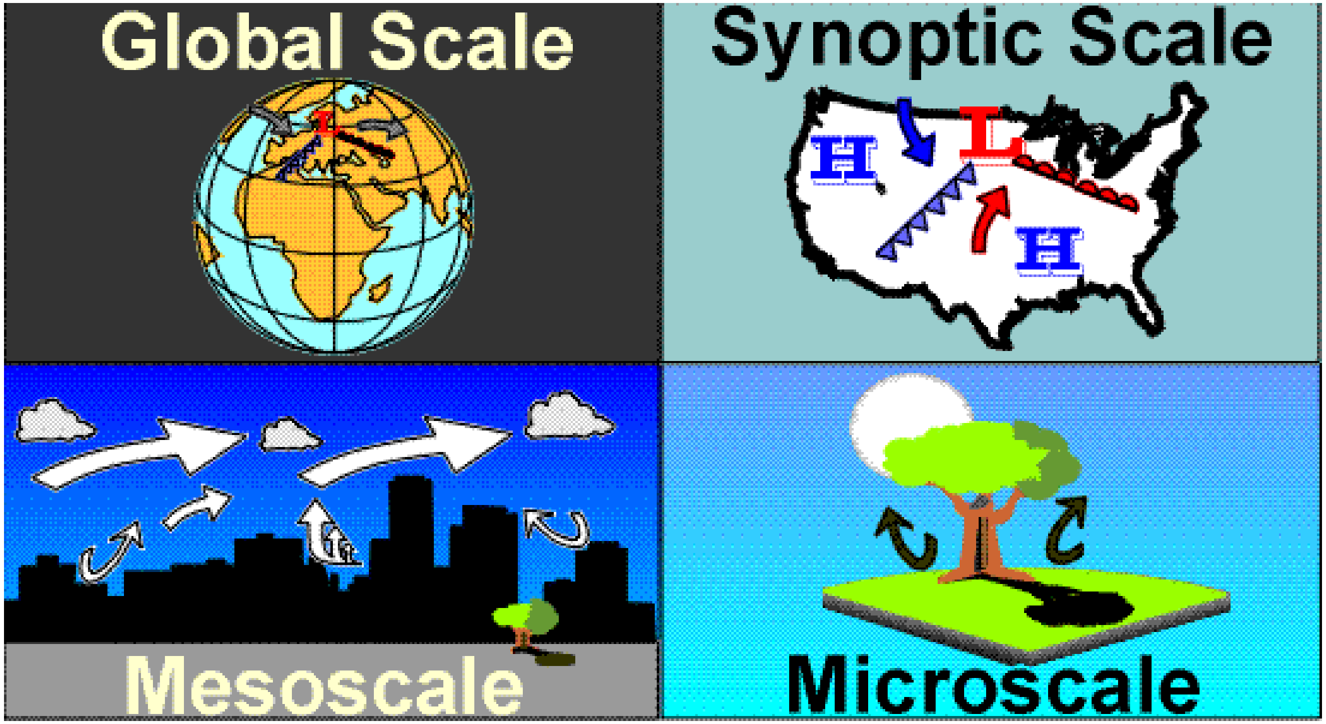
Microscale

Static example: Dali -Lincoln portrait-

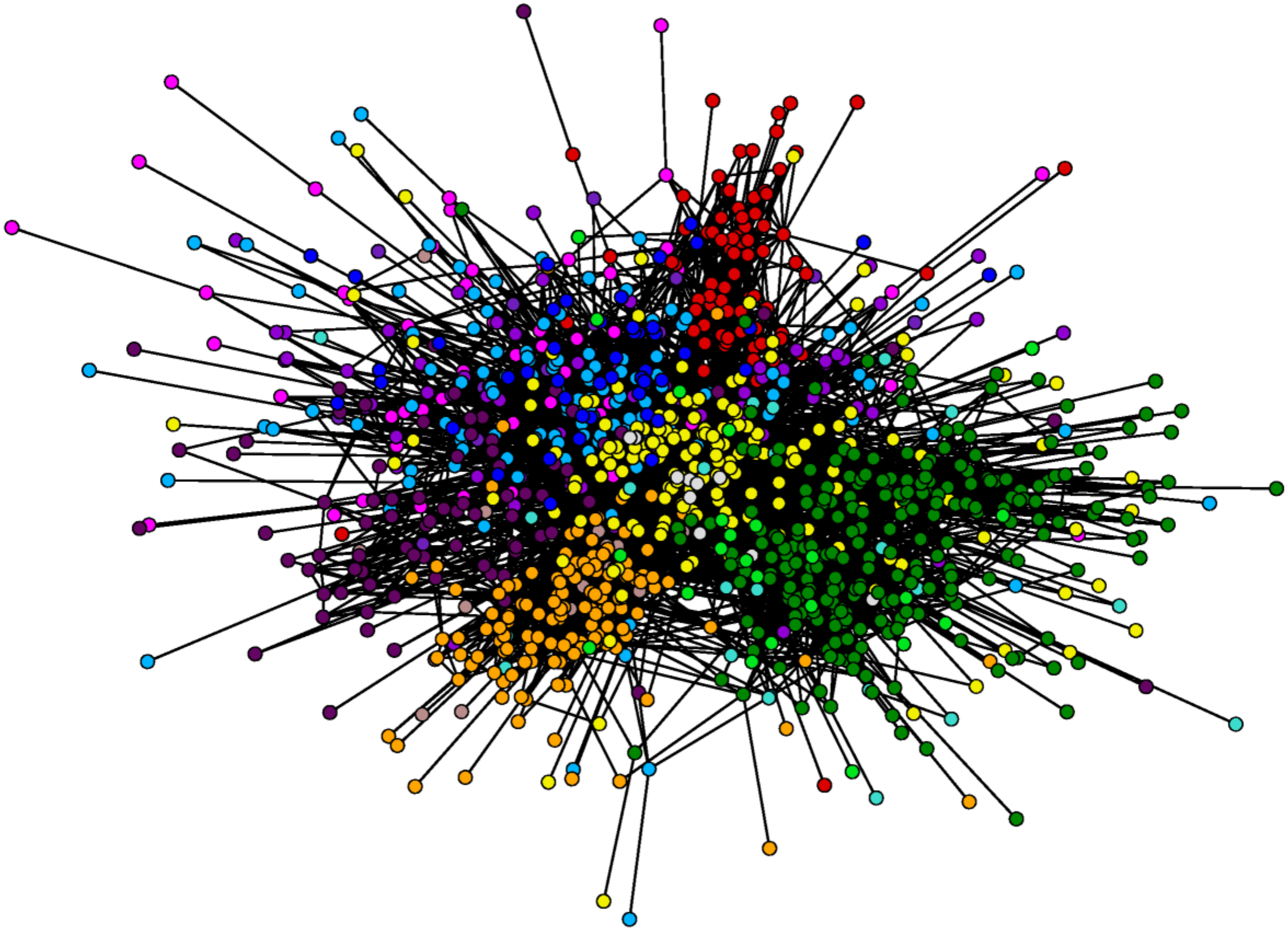


Mesoscale

Dynamic example: Weather prediction models

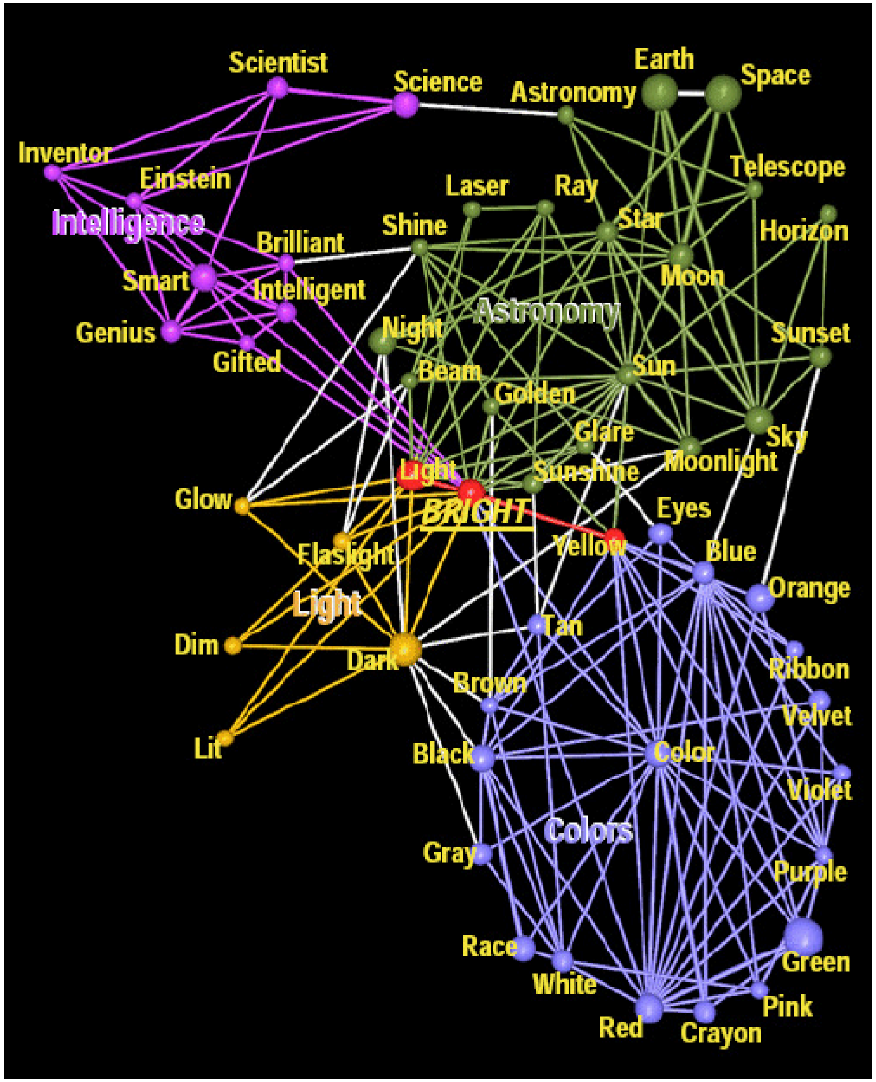


Complex networks of interactions

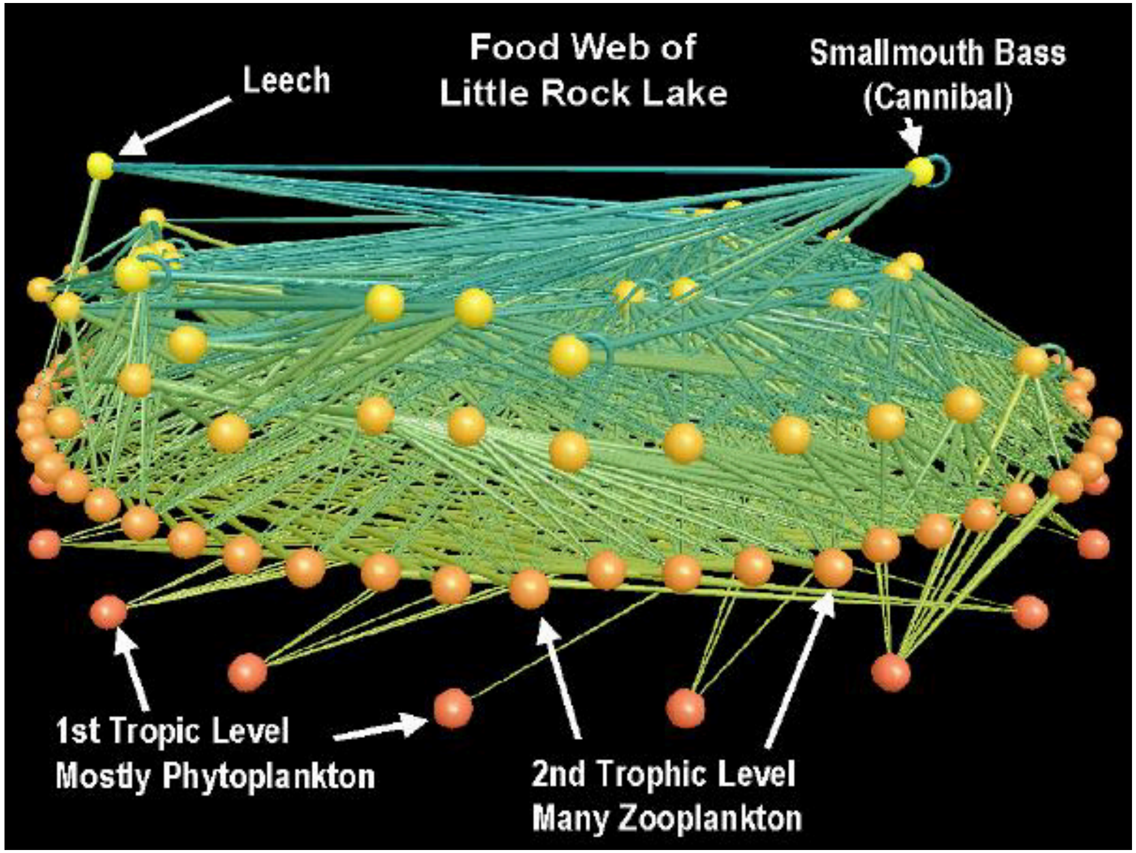


URV (Tarragona, Spain) e-mails inter-exchange network

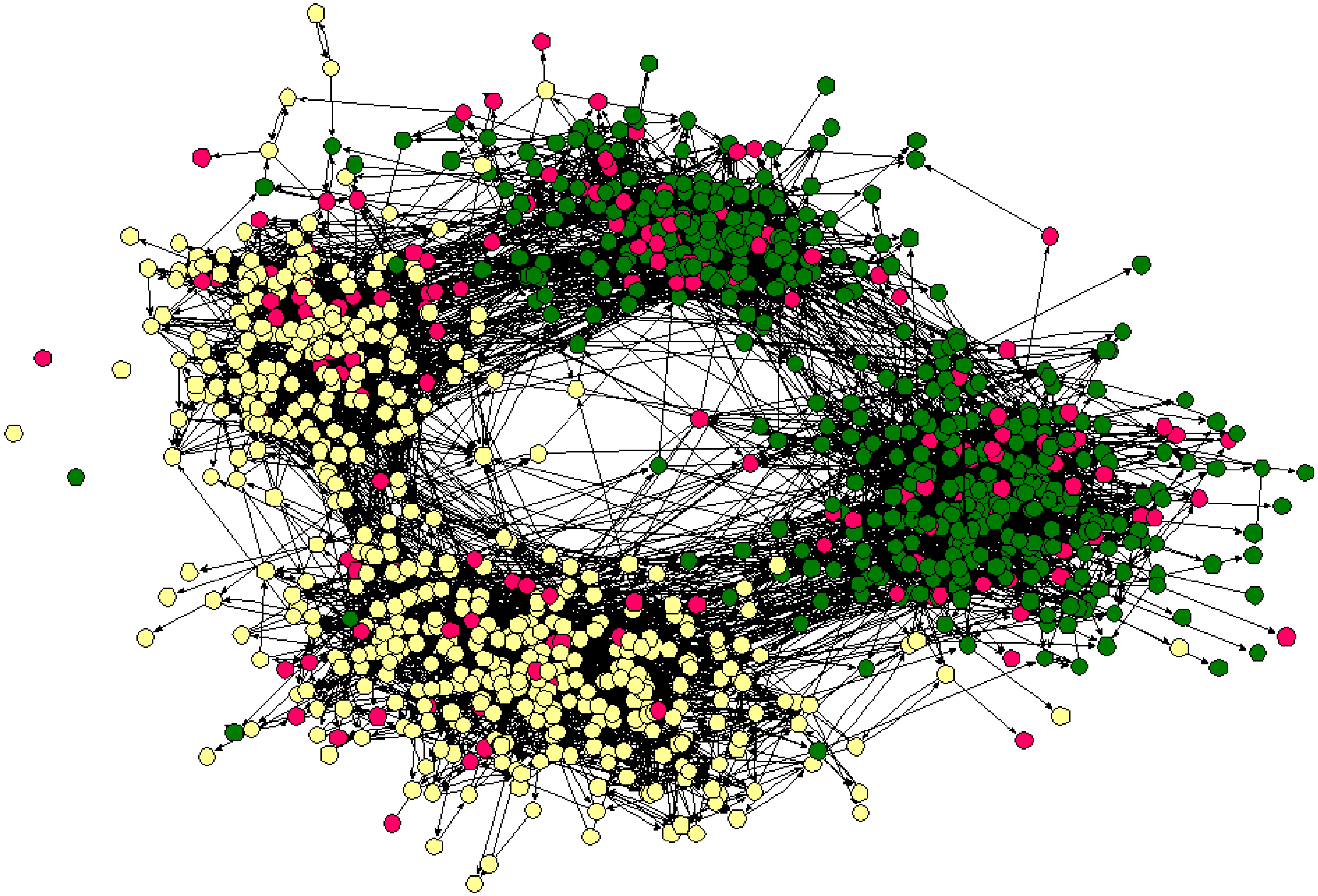
Word-association network



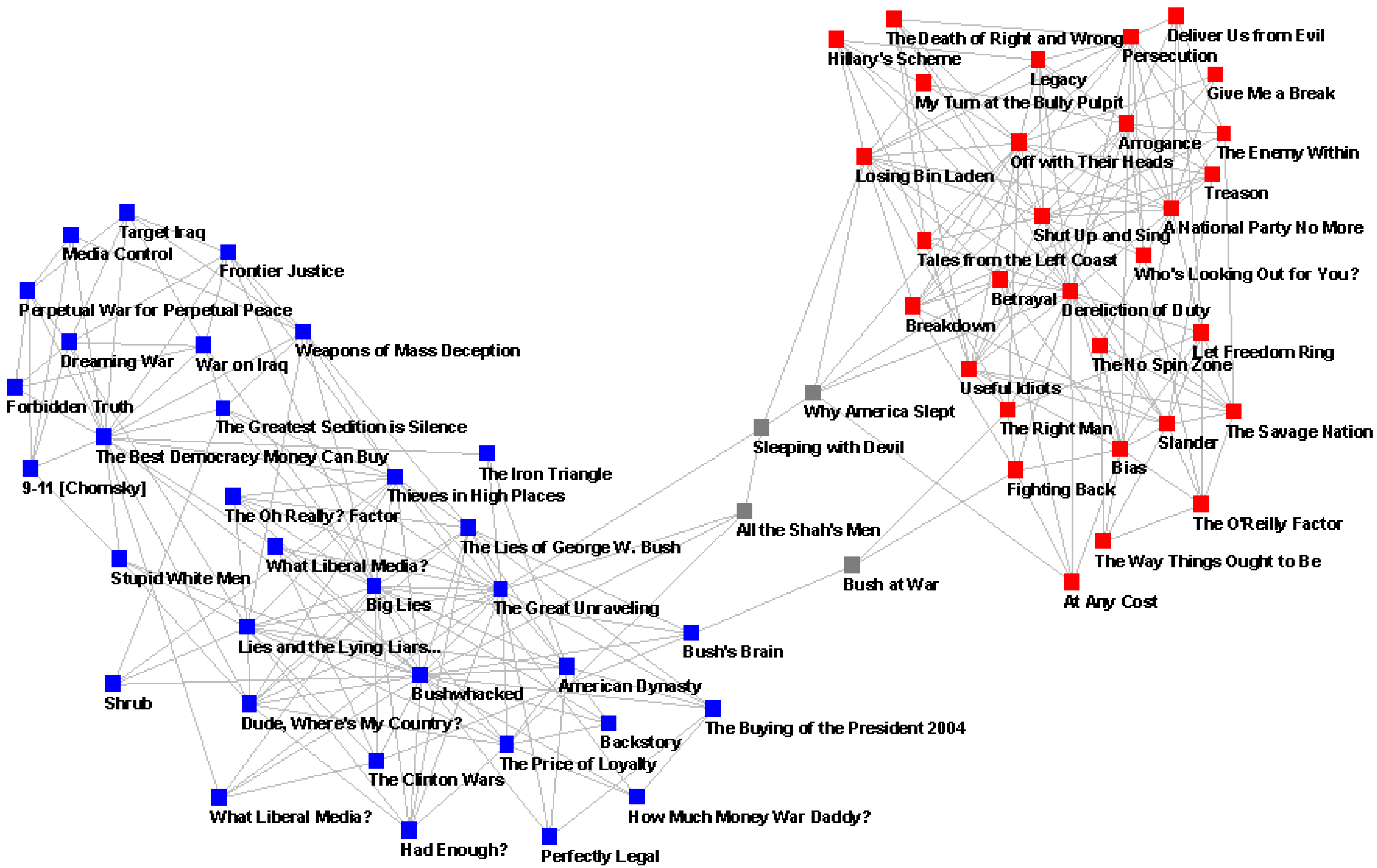
Food-web network



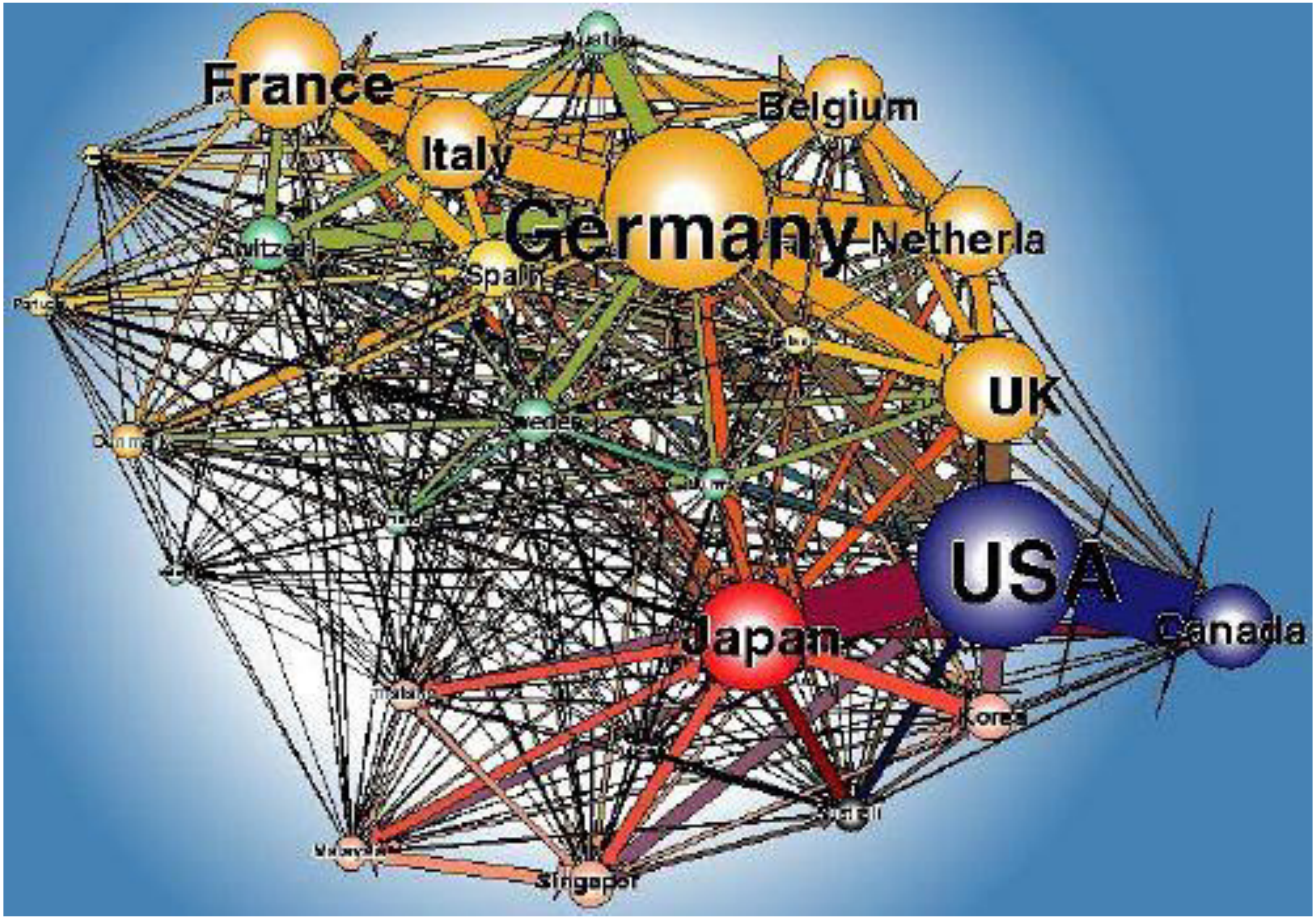
High-school friendship network



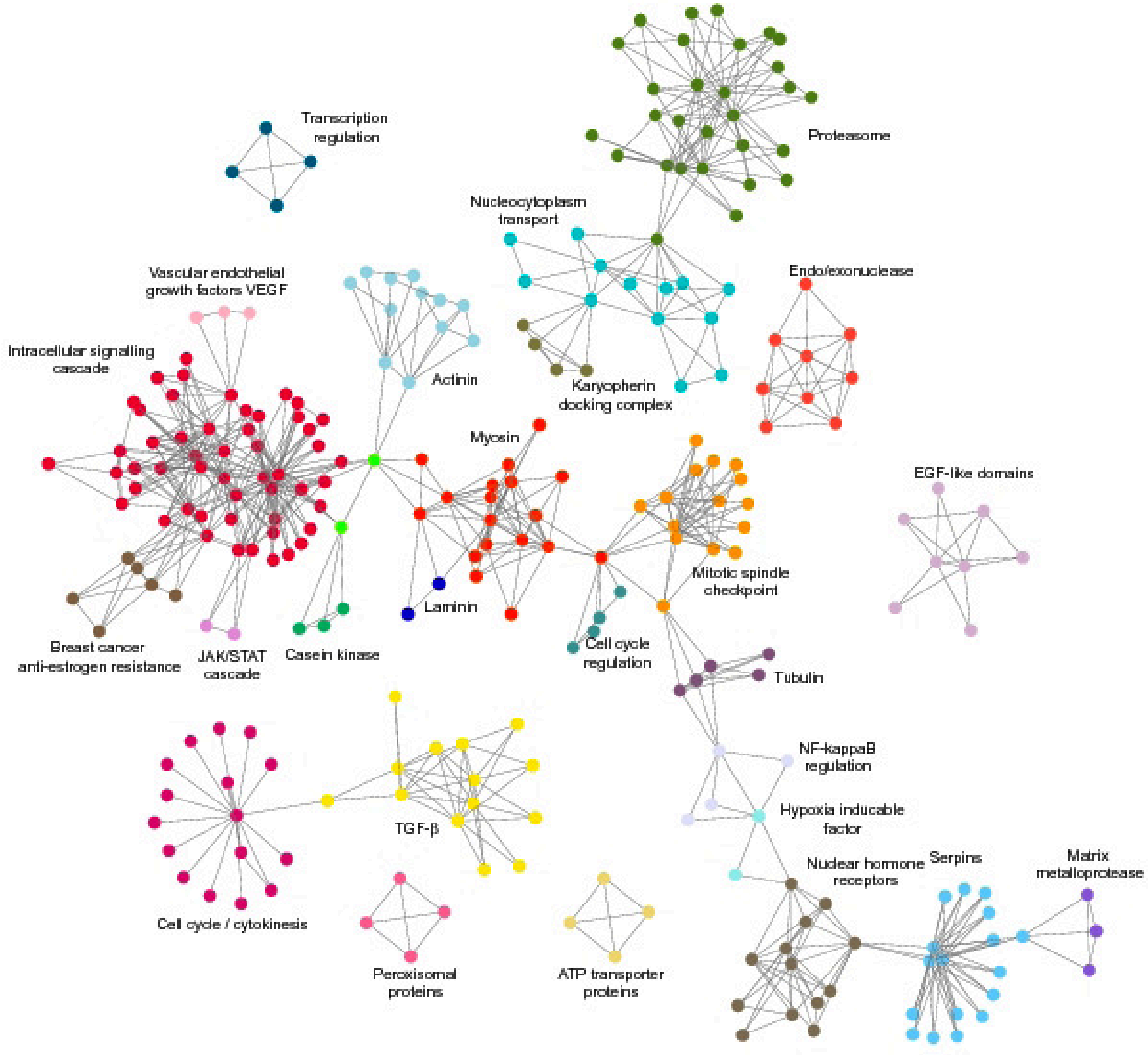
Books on politics network



World Trade Web network



Protein-protein interaction network



Different analysis at different scales

- Microscale: properties of individual nodes or motifs
- Macroscale: statistical properties of the whole network
- Mesoscale: sub-structure of networks "Communities" (also k-cores, n-cliques and n-loops)

Why it is interesting the mesoscale?

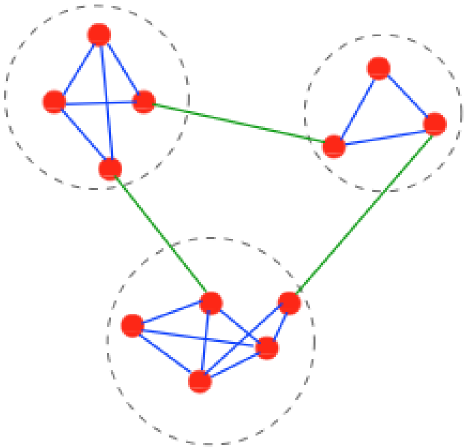
- Similitude with usual coarse-graining in physics
- New statistical properties at this level of description
- Promising way to unravel the interplay between topology and dynamics

Community Detection

- Real complex networks present internal structure (clusters, modules, groups, communities) that can be identified as denser subgraphs within the whole network
- Behind these sub-structures there is information about the functionality of the system that we would like to discover
- The automatic detection of communities is a difficult problem

The problem with the definition of module

- The first problem we face, to determine the sub-structure of networks, is its very definition
- What is a module or community?



Different definitions of modules

- Strong: Every node within a module satisfy the relation k_in>k_out
- Weak: For all nodes in a module the \sum k_in > \sum k_out
- Modules are cliques
- Modules are nodes reachable at distance X
- Etc....

Implicit definition of communities

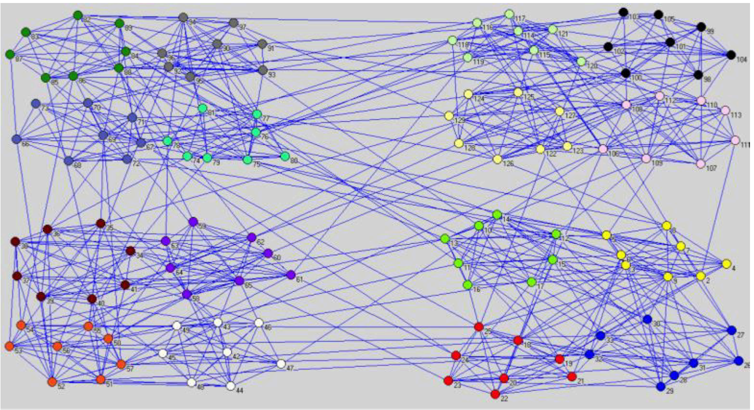
- Modules are the result of maximizing the modularity function (Newman and Girvan)
  - Given a certain disjoint partition of the network, compute the probability of having links between the nodes within a group, minus the same probability if nodes were connected at random but preserving its degree

$$Q = \frac{1}{2m} \sum_{ij} \left(A_{ij} - \frac{k_i k_j}{2m} \right) \delta(C_i, C_j).$$

What is the role of topological structure in the dynamics?

- Syncronization in complex networks
- Our approach:
  - Topology: mesoscale description, community structure
  - Dynamics: transient towards synchronization

Transient towards synchronization

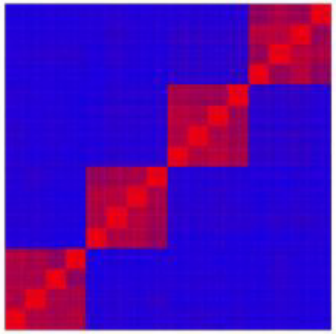


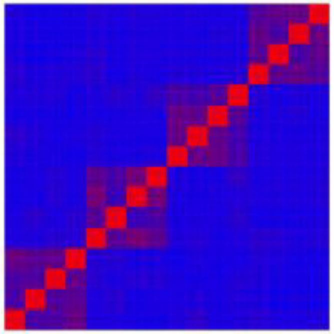
Kuramoto's model

$$\frac{d\ {}_i}{dt} = K \quad {}_{j \quad (i)} \sin(\ {}_j \qquad {}_i)$$

Correlation matrix

$$\rho_{ij}(t) = < cos(\theta_i(t) - \theta_j(t)) >$$





All hierarchical partitions are revealed through time !

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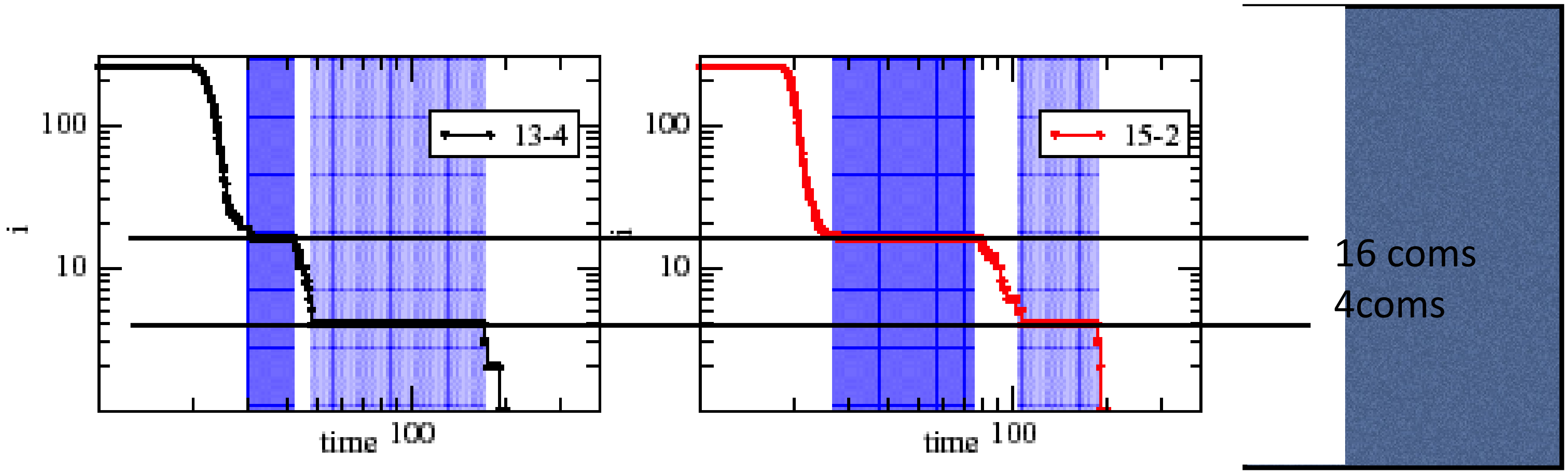
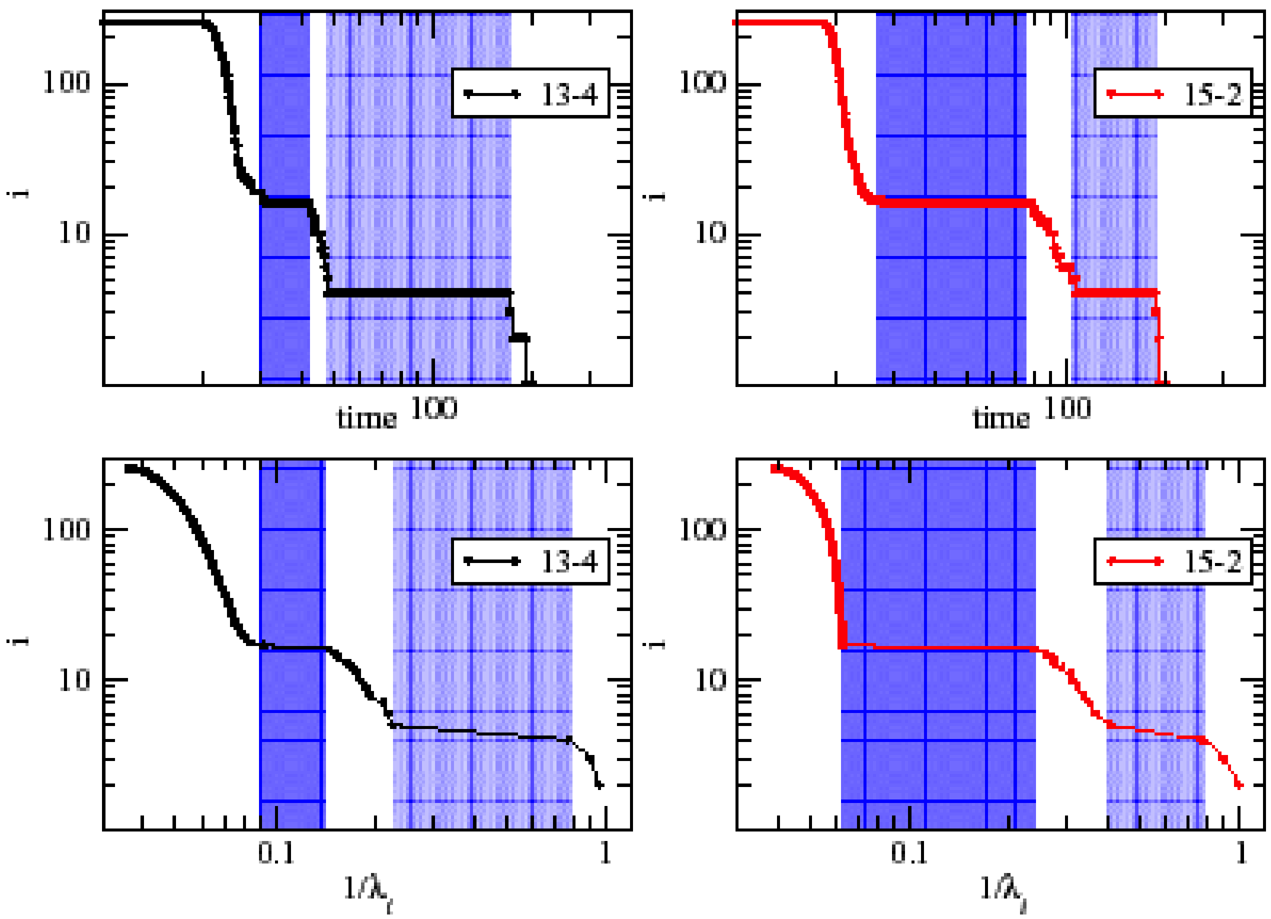
Analysis of the correlation matrix

Dynamic connectivity matrix

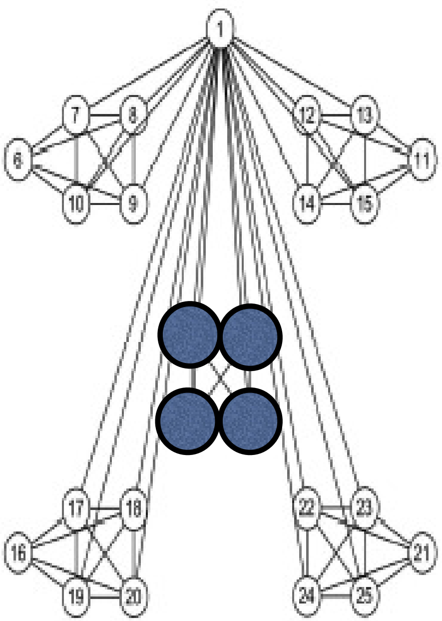
$$\mathcal{D}_T(t)_{ij} = \begin{cases} 1 & \text{if } \rho_{ij}(t) > T \\ 0 & \text{if } \rho_{ij}(t) < T \end{cases}$$

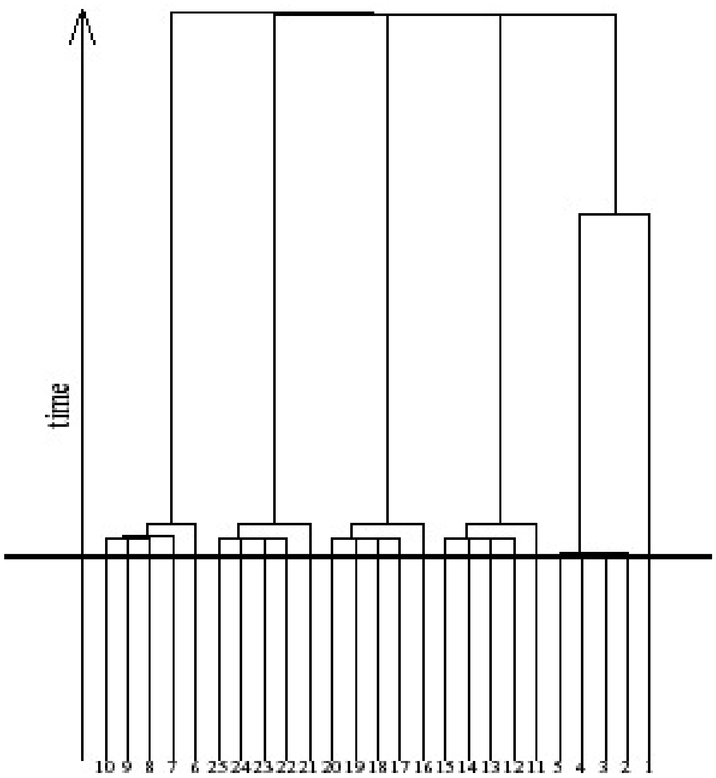


The connection between topology and dynamics



The connection between topology and dynamics





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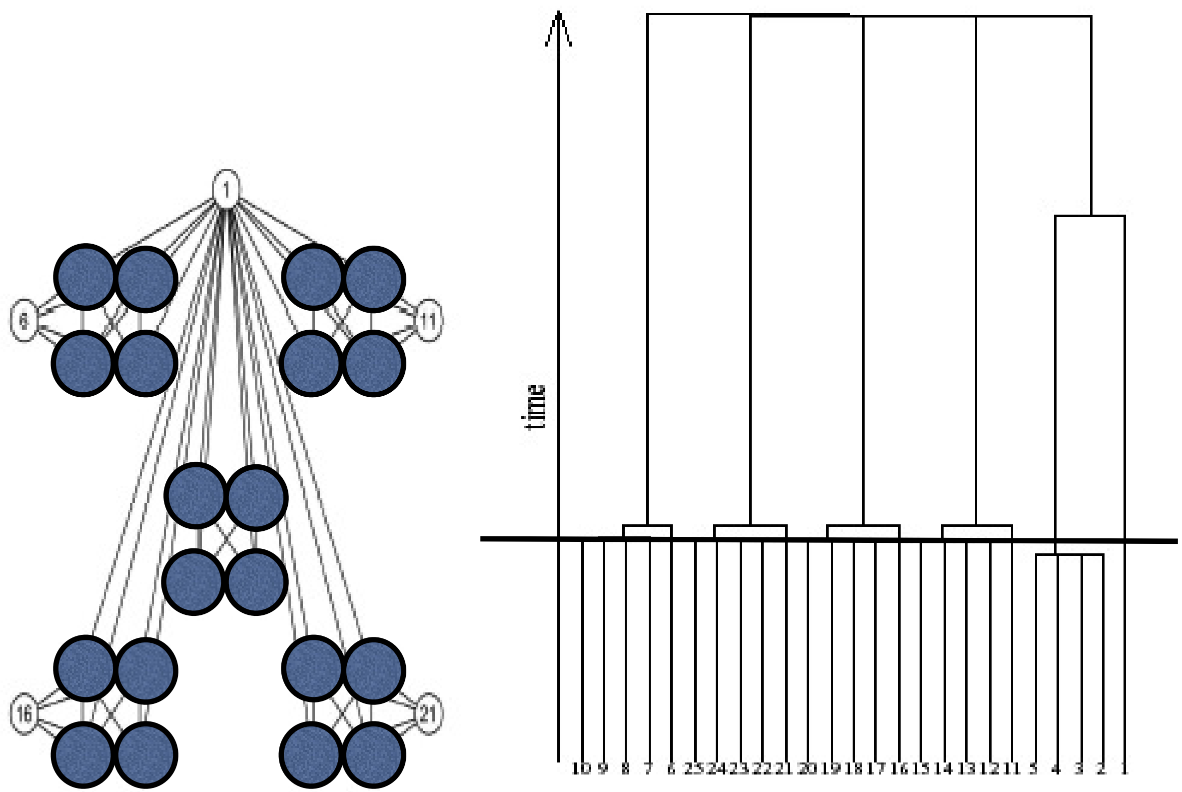
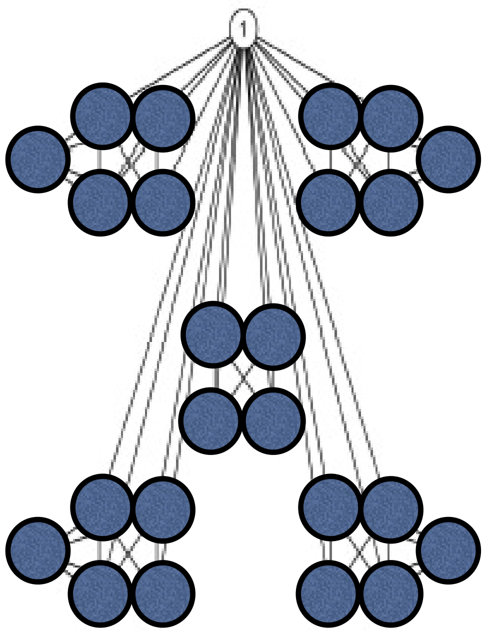
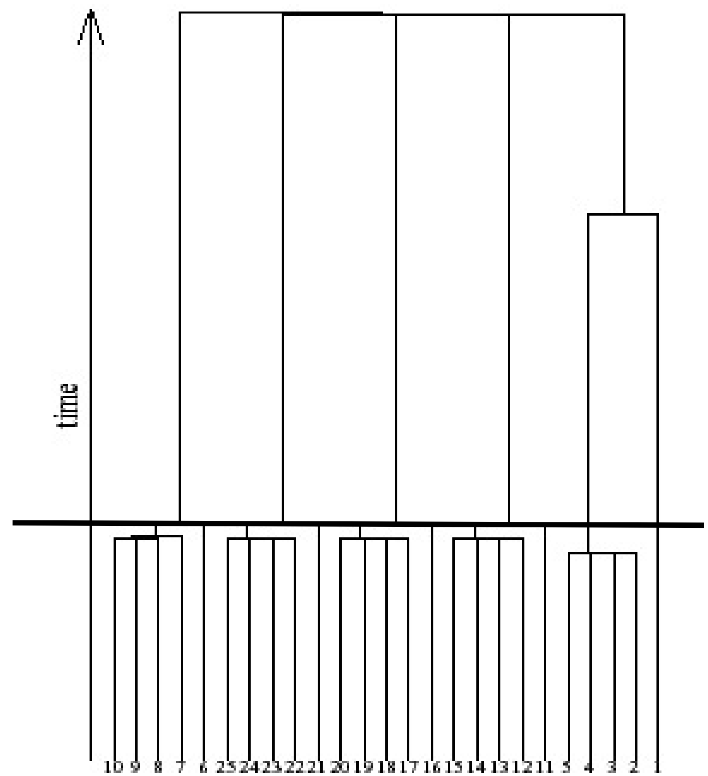
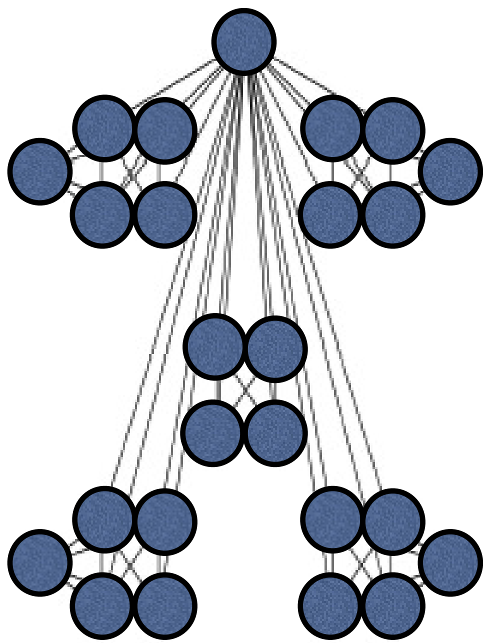


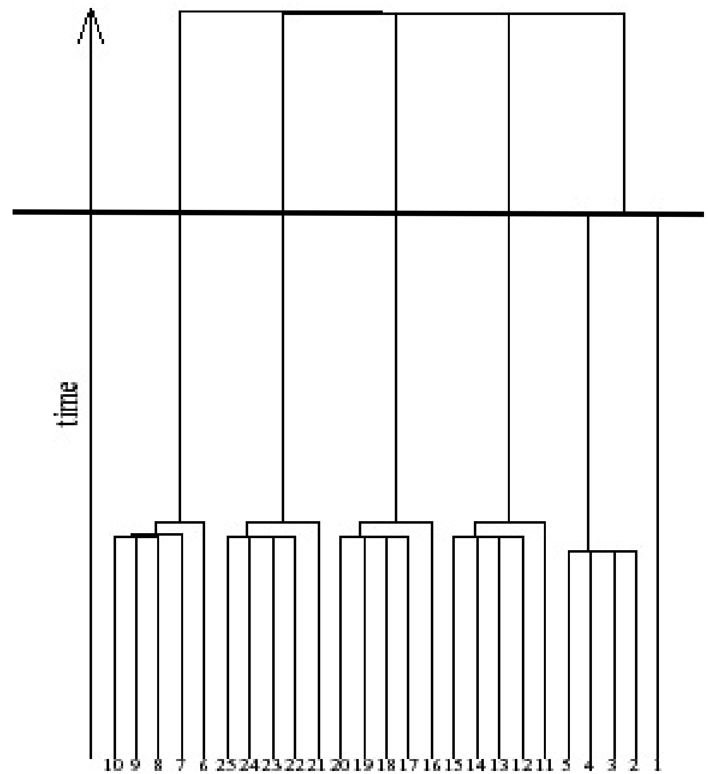
The connection between topology and dynamics





The connection between topology and dynamics





Understanding the problem

Linearized version of the problem

$$\frac{d\theta_i}{dt} = -k \sum_j L_{ij}\theta_j$$

Solution in terms of normal modes

$$\varphi_i(t) = \sum_j B_{ij}\theta_j = \varphi_i(0)e^{-\lambda_i t}$$

Understanding the problem

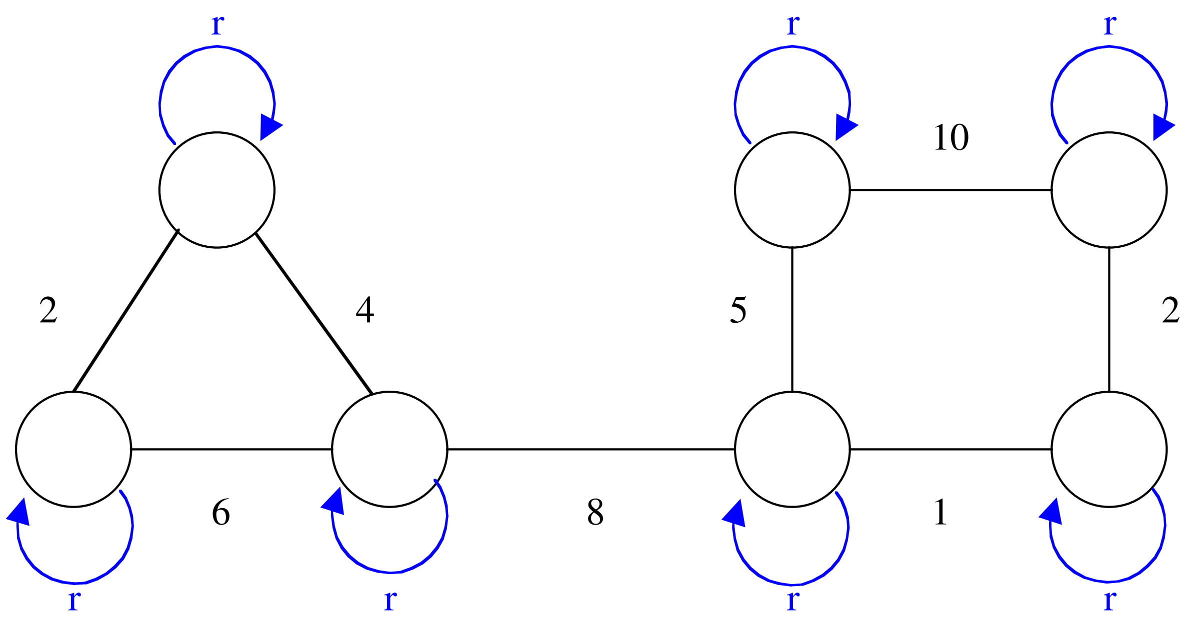
- If we rank the equations from fastest to lower modes we observe a cascade of synchronization restrictions
- The largests eigenvalues involve “topological average” equations
- Similar eigenvalues involve closed equations that synchronize groups of oscillators at the same time scale

How to unravel the mesoscale from the topology?

- Idea:
- To provide each node with a global magnitude that will tune its strength at different intermediate topological scales without affecting its structural topology, i.e. preserving P(k), vertex-vertex correlations and spectrum
- Find community structure at each resolution level. We overcome the resolution limit first found by Fortunato & Barthelemy

How to unravel the mesoscale from the topology?

We introduce a resistance "r" of nodes to become part of a group



Some notation

Weighted matrix

$$w'_{ij} = \begin{cases} w_{ij} & \text{if} \quad i \neq j\,, \\ r & \text{if} \quad i = j\,. \end{cases}$$

Node strength

$$w'_i = \sum_j w'_{ij} = w_i + r$$

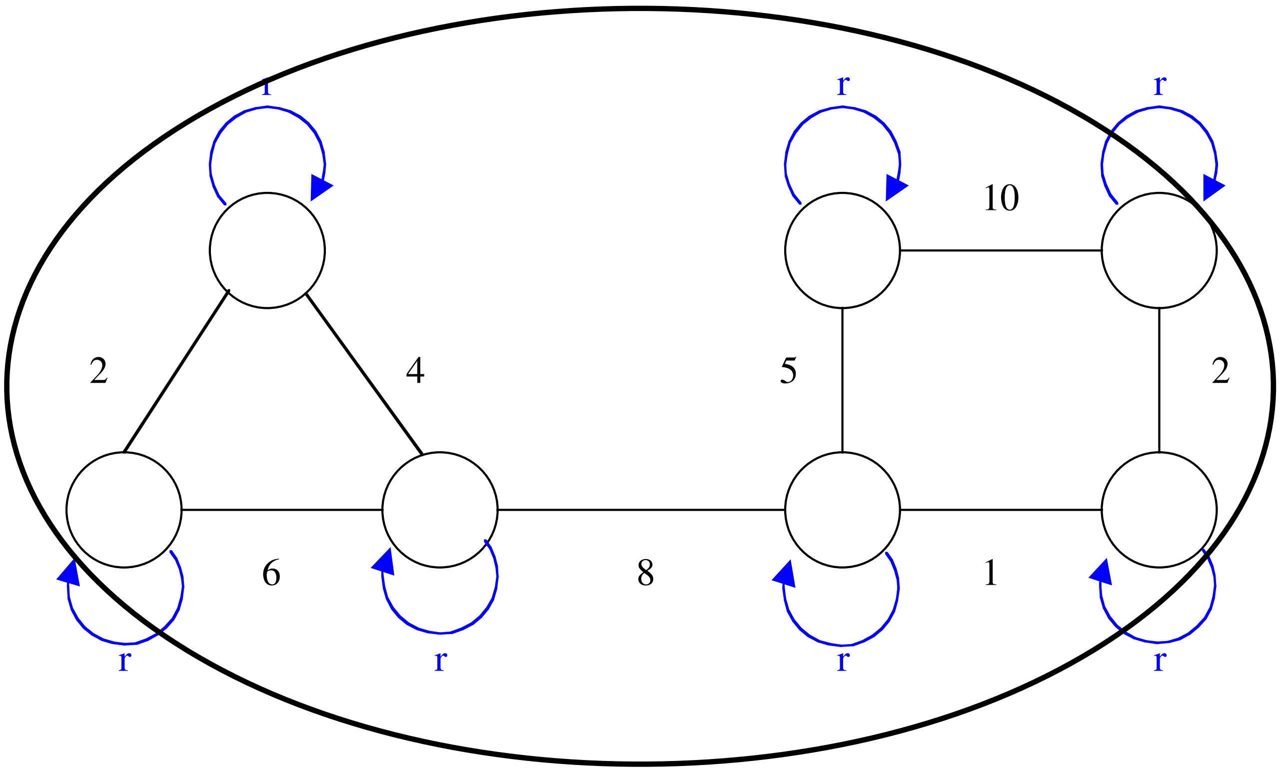
Total strength

$$2w' = \sum_i w'_i = 2w + Nr$$

$$Q(r) \;\; = \;\; \frac{1}{2w'} \sum_i \sum_j \left(w'_{ij} - \frac{w'_i w'_j}{2w'} \right) \delta(C_i, C_j)$$

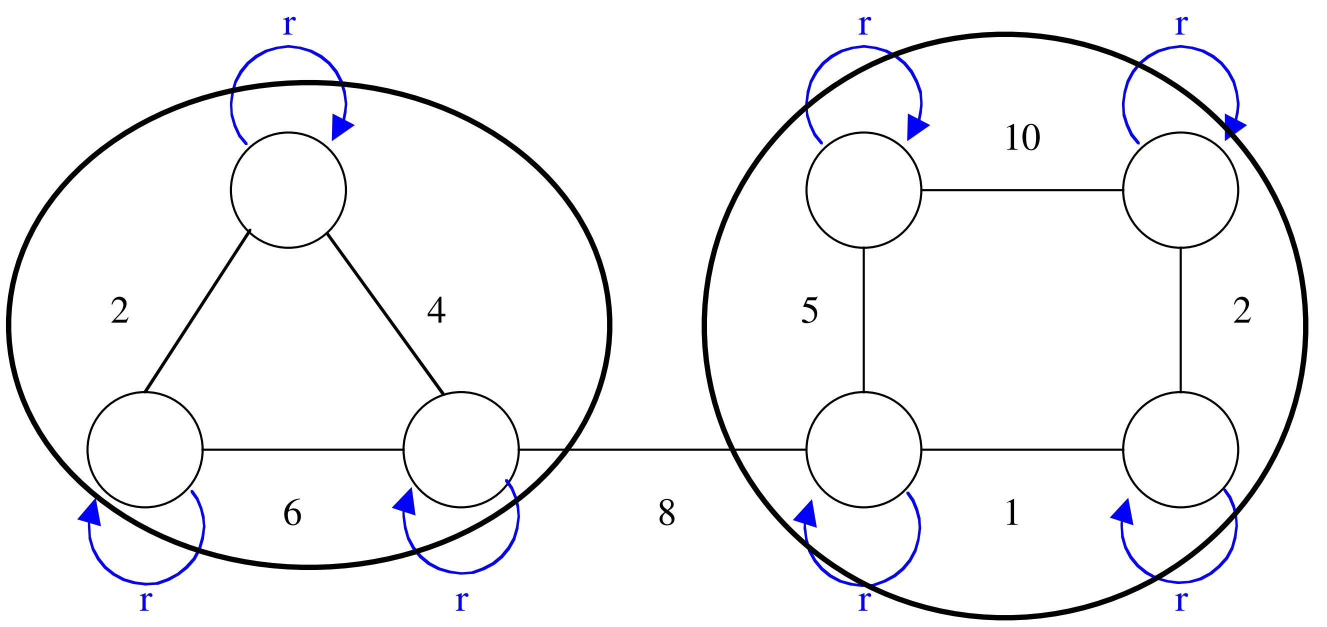
Modularity, original formulation

Community structure at different values of r



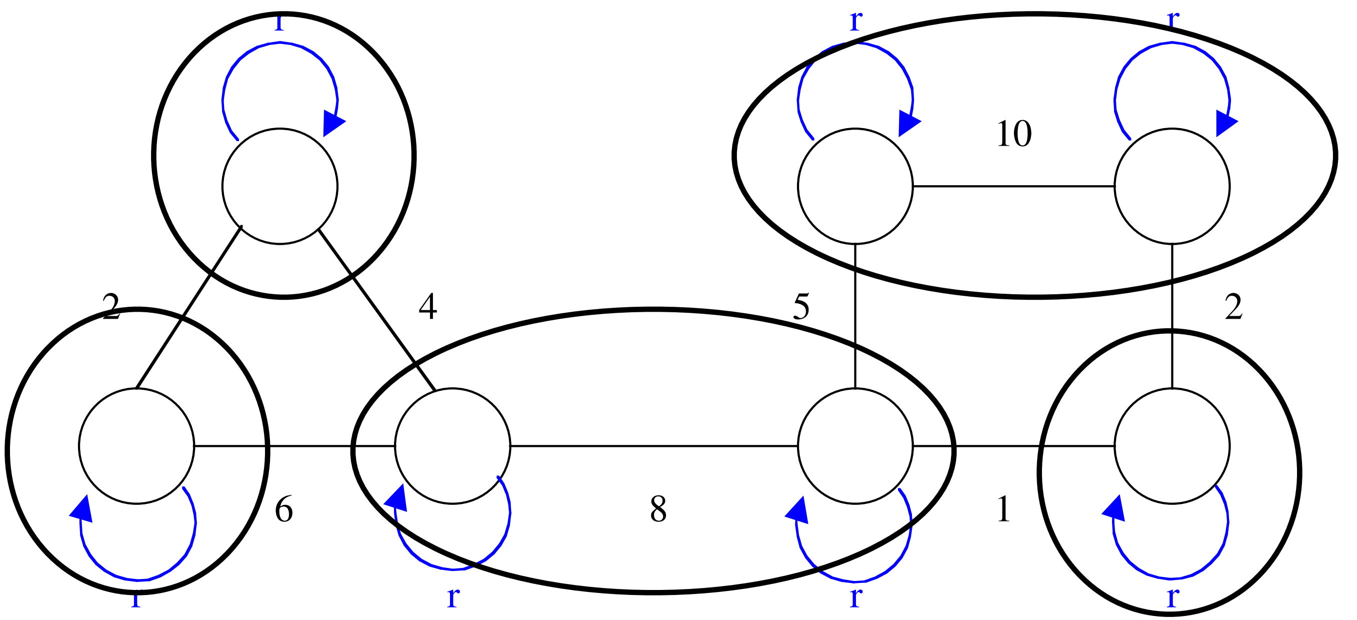
r_min

Community structure at different values of r



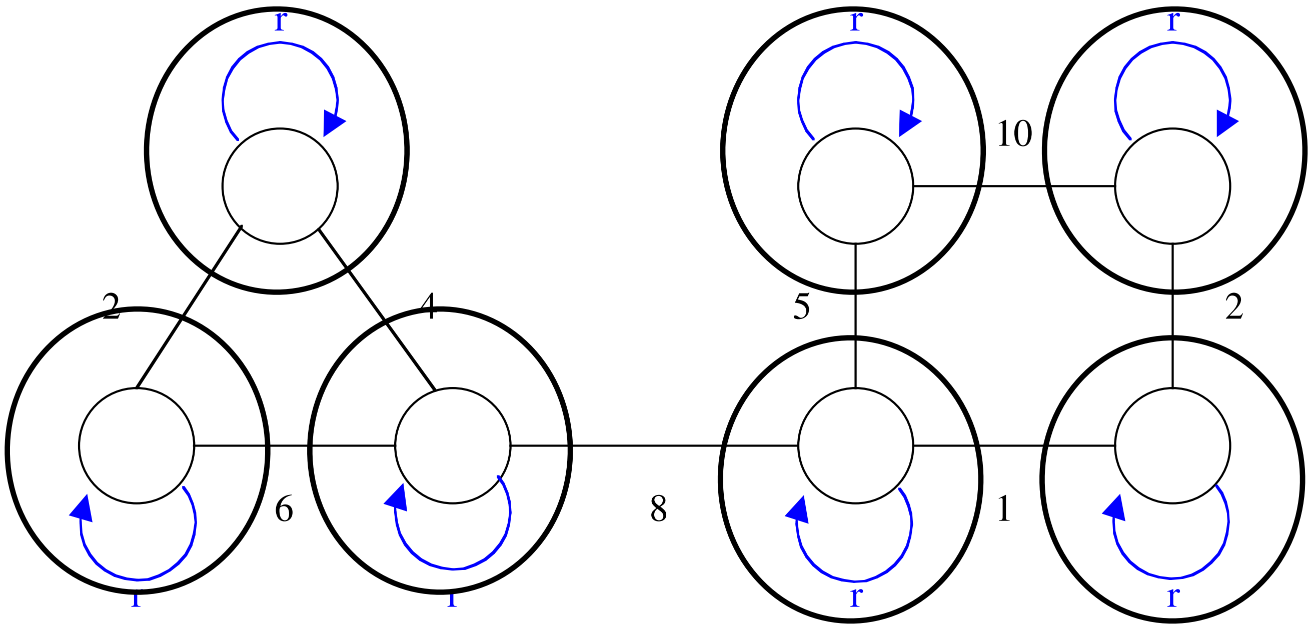
r1 intermediate

Community structure at different values of r



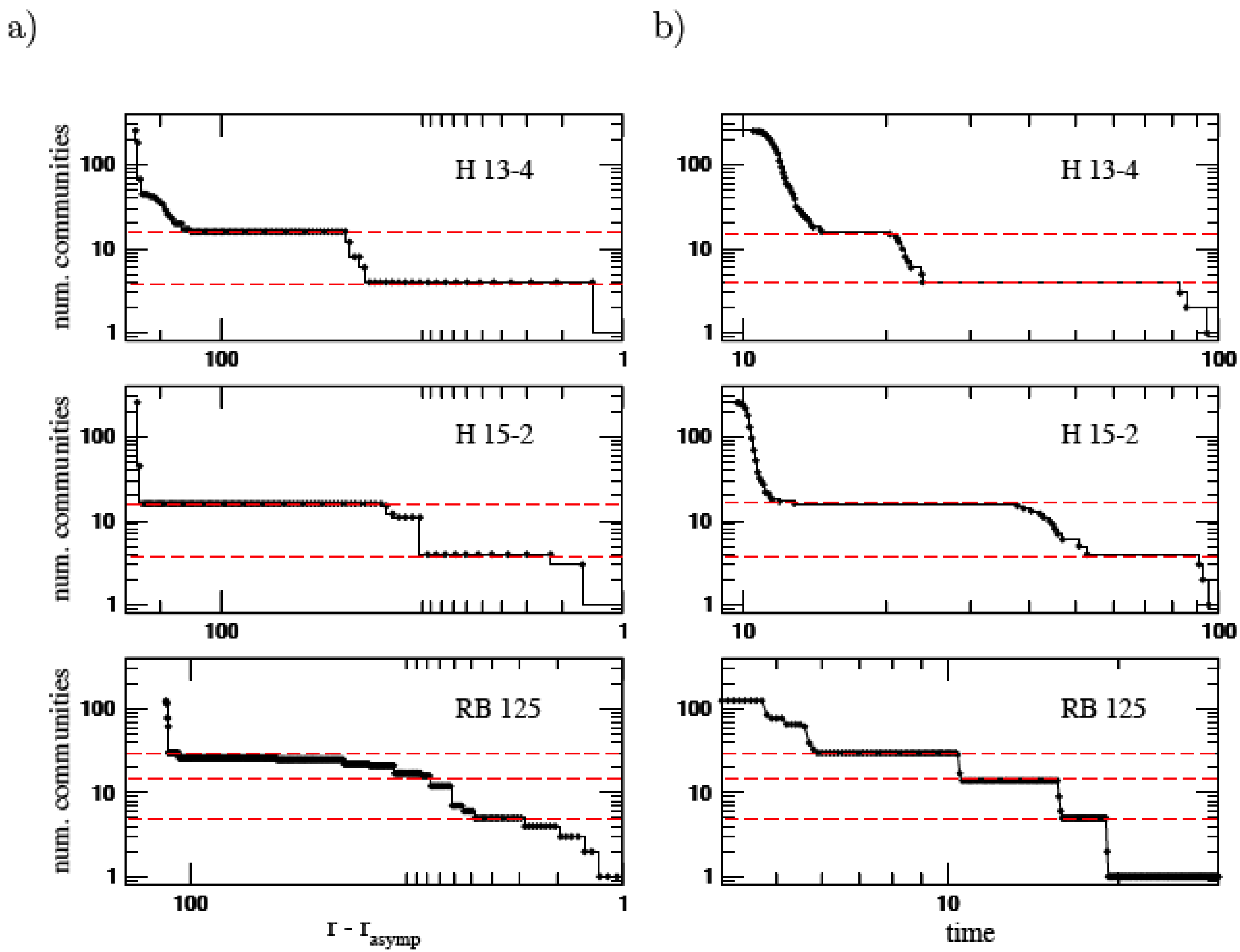
r2 intermediate

Community structure at different values of r

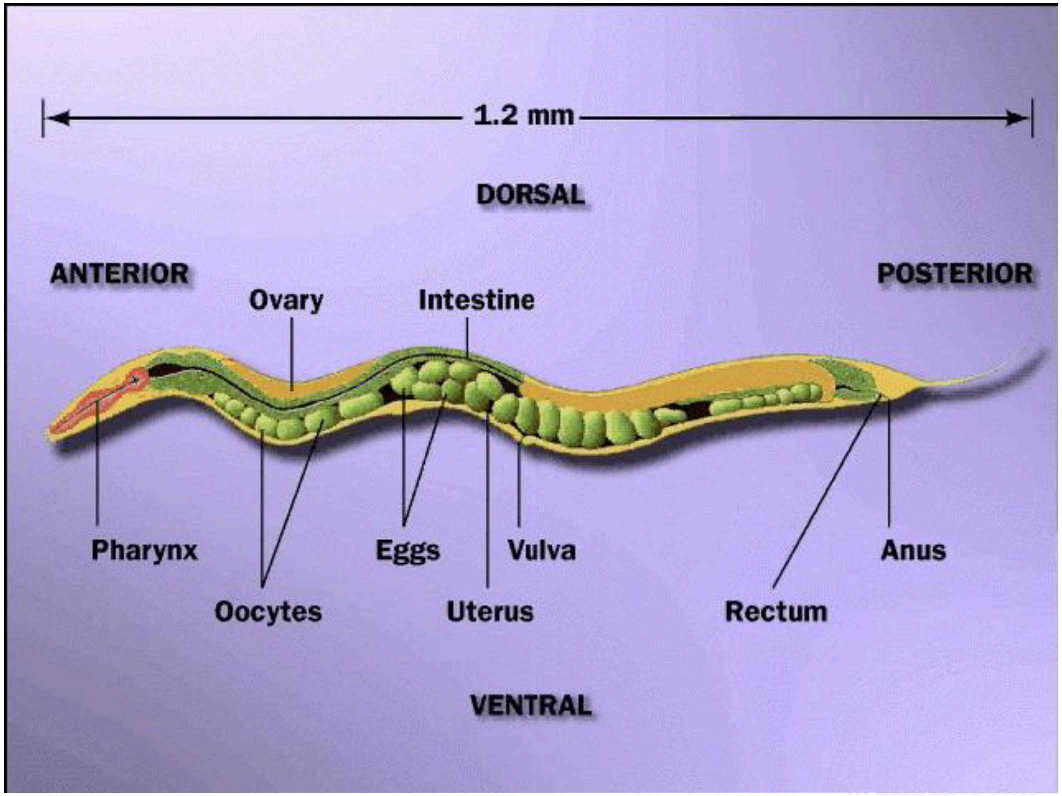


r_max

Comparing the dynamic and static mesoscale



Applications: mesoscale of C.elegans neural network

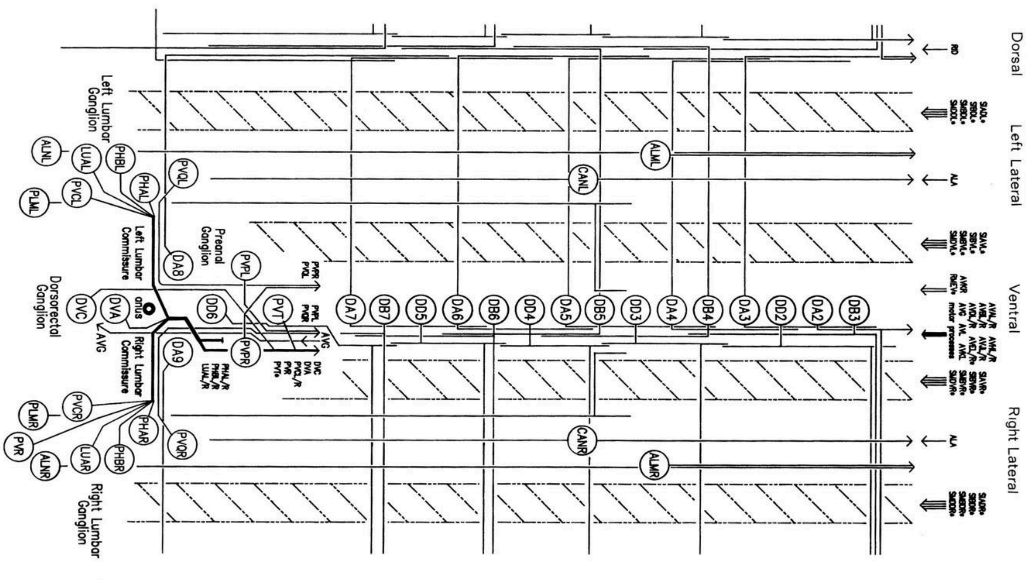


Why to study C. elegans neural network?

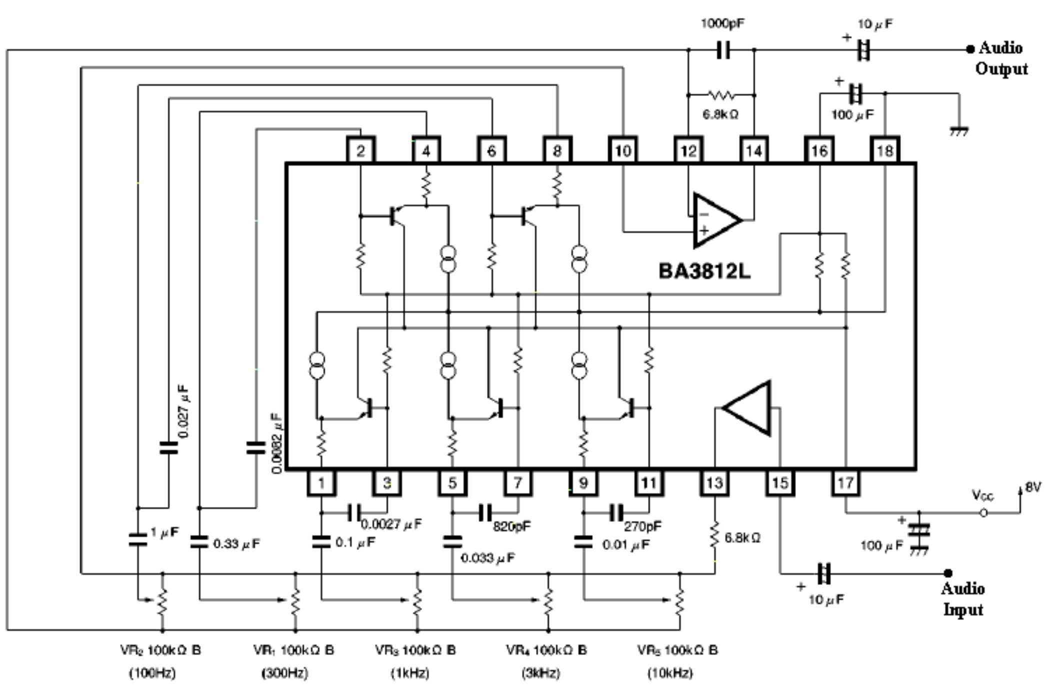
- C. elegans is the most studied living organism in history, the detail of knowledge is exhaustive
- Neural system not plastic, genetically designed
- Still missing the structure-functionality relationship

Kind of reverse engineering

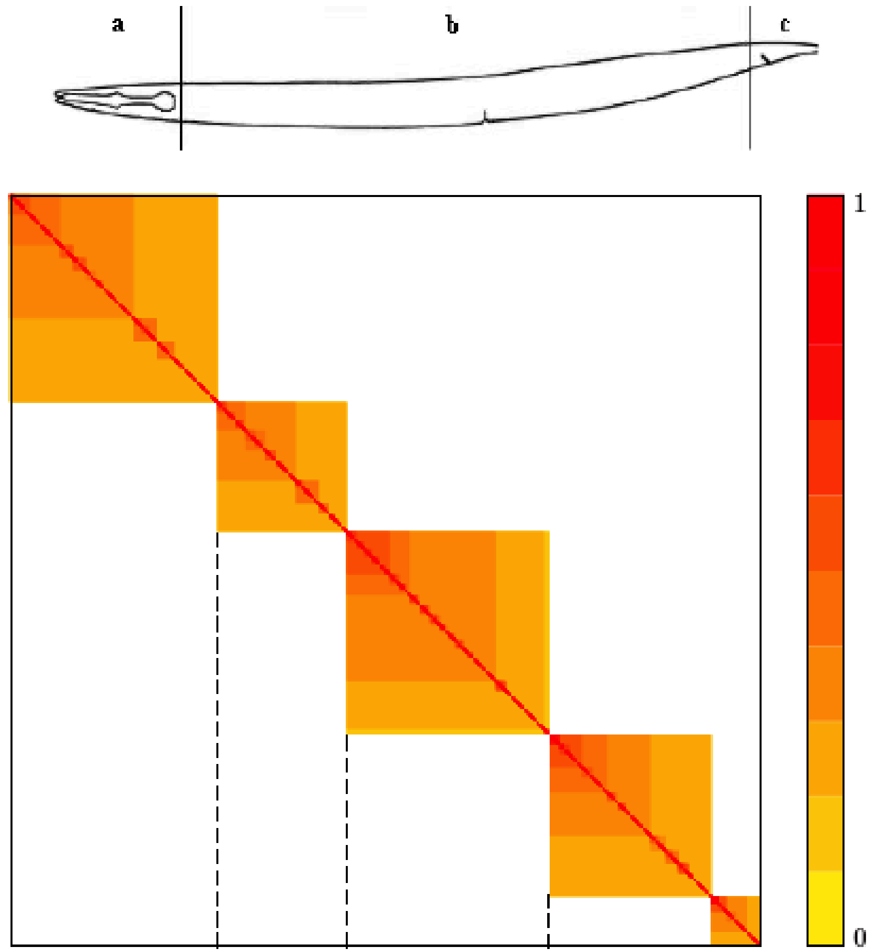
C.elegans neural system



Chip electronic circuit

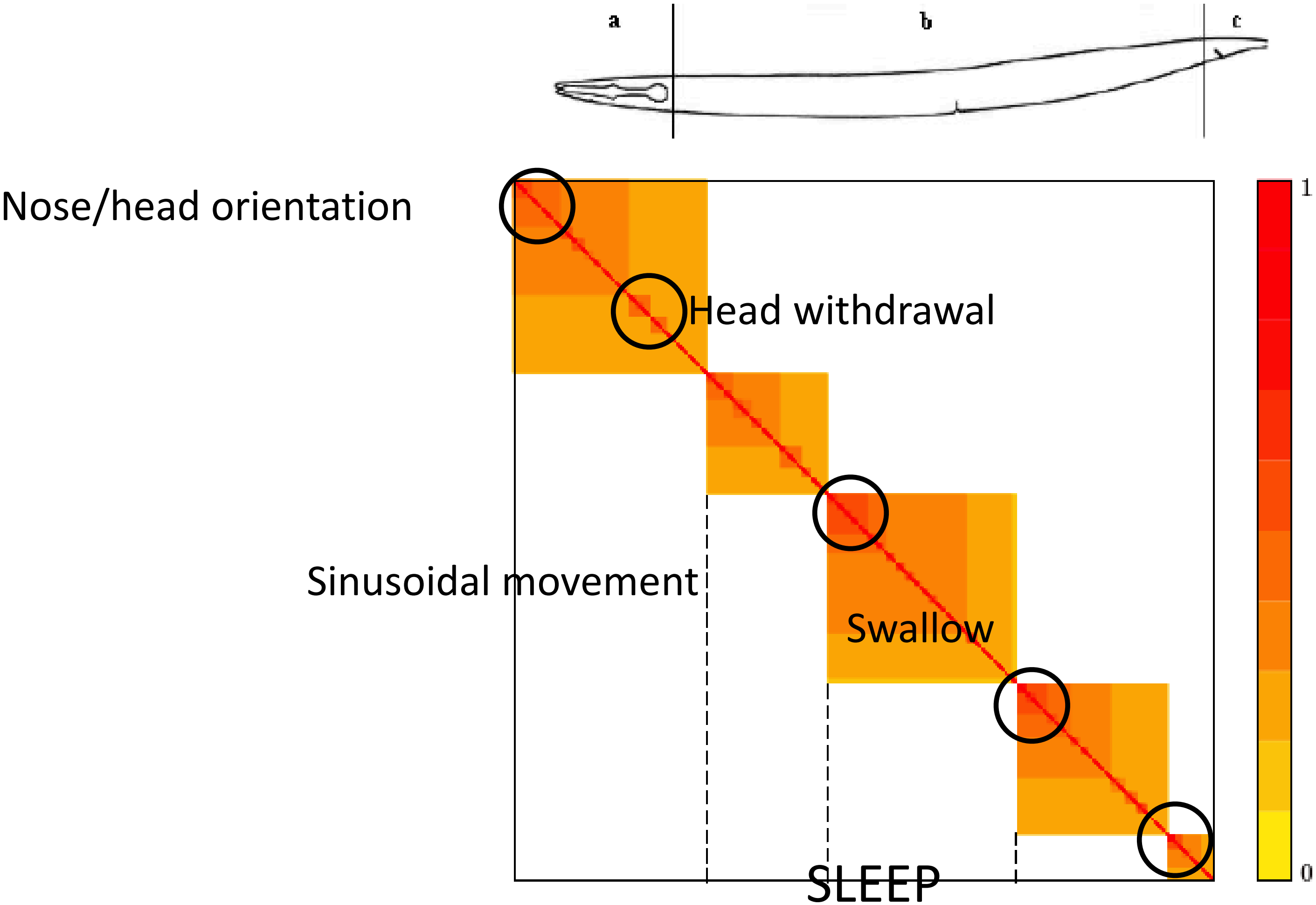


Mesoscale: location correlations



a	96.4 %	92.0 %	15.2 %	40.6 %	36.8 %
b	1.2 %	0.0 %	82.3 %	32.8 %	26.4 %
c	2.4 %	8.0 %	2.5 %	26.6 %	36.8 %

Mesoscale: functional correlations



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c	2.4 %	8.0 %	2.5 %	26.6 %	36.8 %

Applications: Data Clustering

- Clustering is the unsupervised classification of patterns (observations, data items, or feature vectors) into groups (clusters).

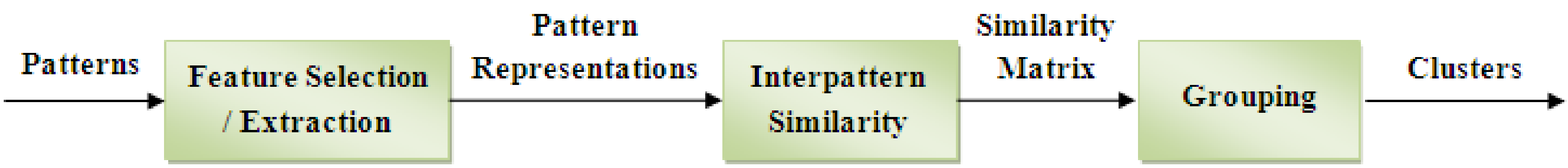
Some applications:

Marketing: finding groups of customers with similar behavior given a large database of customer data containing their properties and past buying records.

WWW: document classification; clustering weblog data to discover groups of similar access patterns.

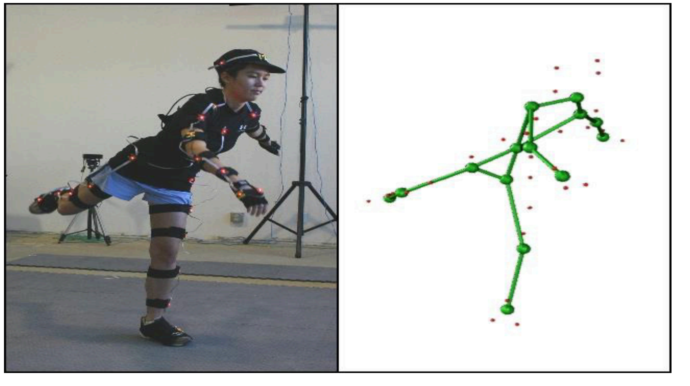
Biology: classification of plants and animals given their features.

Applications: Data Clustering



Feature selection / extraction

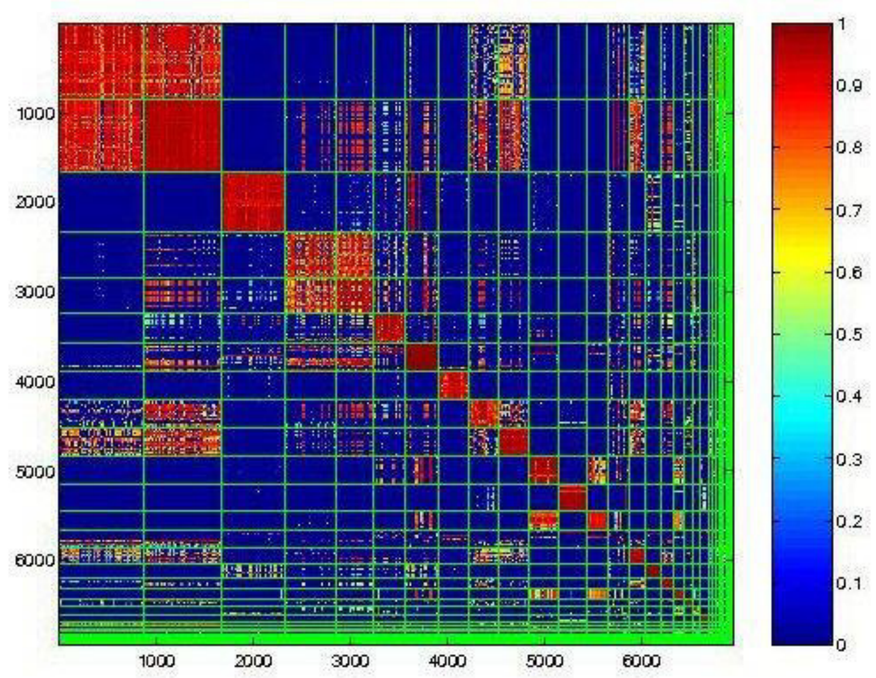
- Stands for the process of selecting the most representative variables for the classification process.
- Many popular search approaches use heuristics which iteratively evaluates a candidate subset of features, then modifies the subset and evaluates if the new subset is an improvement over the old using a scoring metrics.



Interpattern Similarity

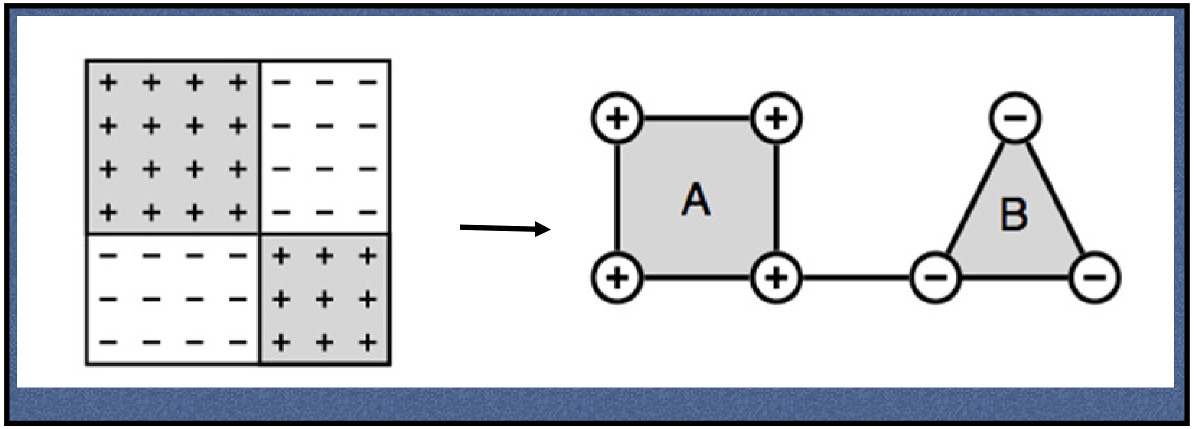
- Similarity Matrix: A matrix of scores expressing the similarity between two data points in terms of distances (e.g Minkowski distance).

$$d_p(x_i, x_j) = \left(\sum_{k=1}^{d} \|x_{i,k} - x_{j,k}\|^p \right)^{\frac{1}{p}}$$



Modularity based community analysis

- A representation of a matrix in terms of networks is likely to be analyzed using community detection algorithm.
- Using a reformulation of modularity for signed links



IRIS flower data set
Introduced by Sir R. A. Fisher (1936)

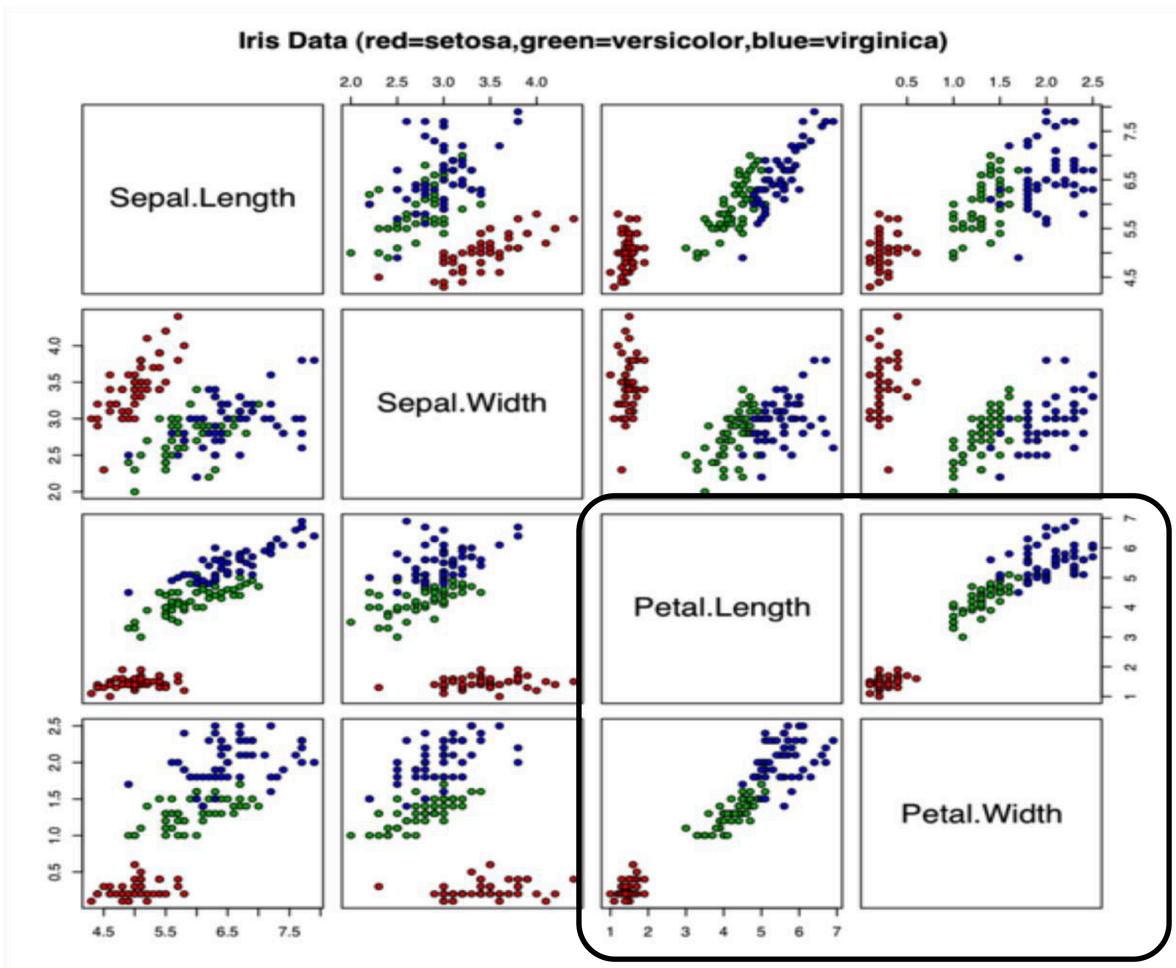
Setosa

Versicolor

Virginica

- Problem: to automatically classify the three types, using a database of:
  - 50 samples of each one, characterized by 4 dimensional vectors, corresponding to length and width of sepal and petal

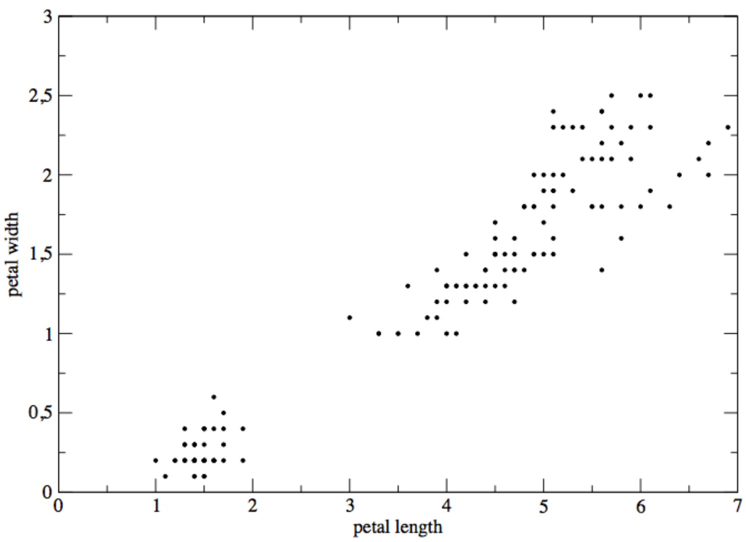
Variable selection

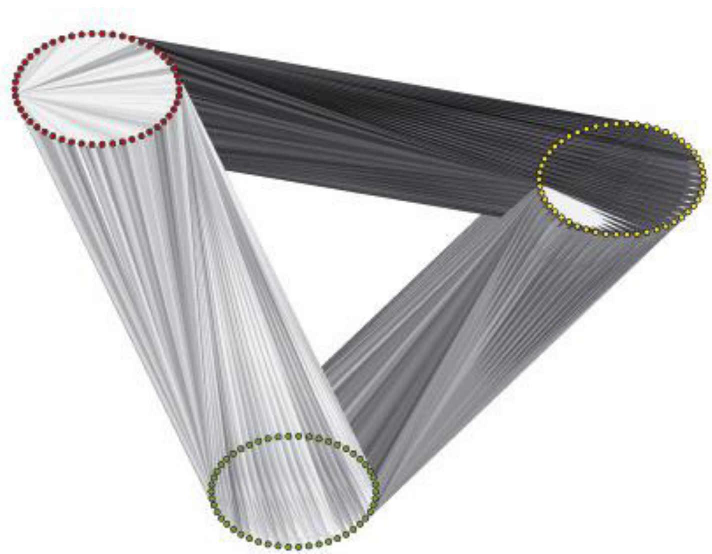


Similarity matrix

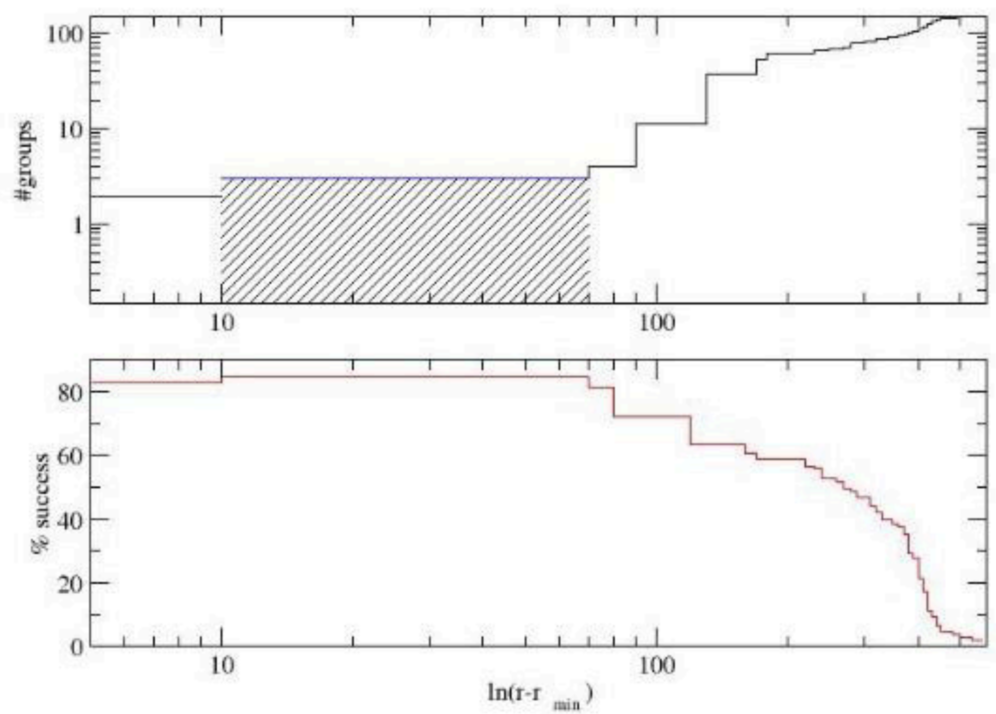
We propose to compute the similarity between data using distances in PCA

$$s_{ij} = d_{ij} - \langle d \rangle$$





Classification results using multi-resolution of modularity



Conclusions

- We have proposed a method to compute the topological mesoscale in complex networks
- The method provide with substructures compatible with those meta-stable patterns observed in the synchronization dynamics
- The substructures obtained in the C.elegans case study allow for a tentative classification of structure-function relations
- The use of community analysis in data clustering is a promising tool.

http://deim.urv.cat/~aarenas

References

Mesoscopic analysis of networks: applications to exploratory analysis and data clustering C. Granell, S. Gómez and A. Arenas, Chaos: An Interdisciplinary Journal of Nonlinear Science, 21, 016102 (2011)

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Analysis of community structure in networks of correlated data S. Gómez, P. Jensen and A. Arenas, Physical Review E 80, 016114 (2009)

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Diffusion processes on multiplex networks

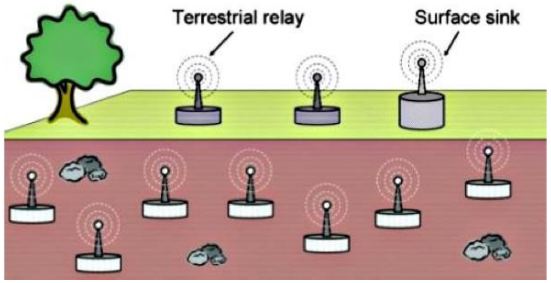
Alex Arenas
Universitat Rovira i Virgili (Tarragona, Spain)

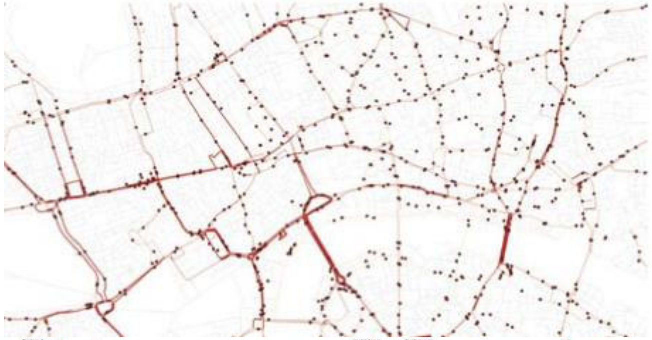
Alephsys

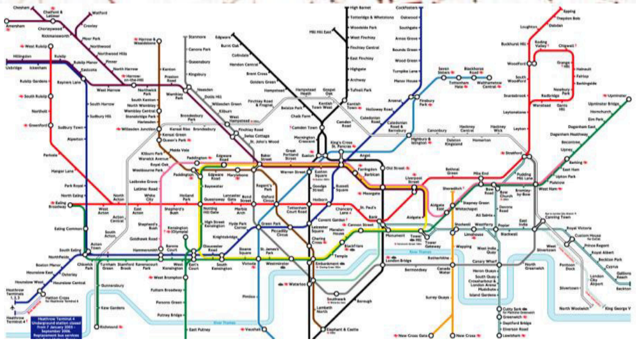
Multiplex networks (EU projects: PLEXMATH, LASAGNE and MULTIPLEX)

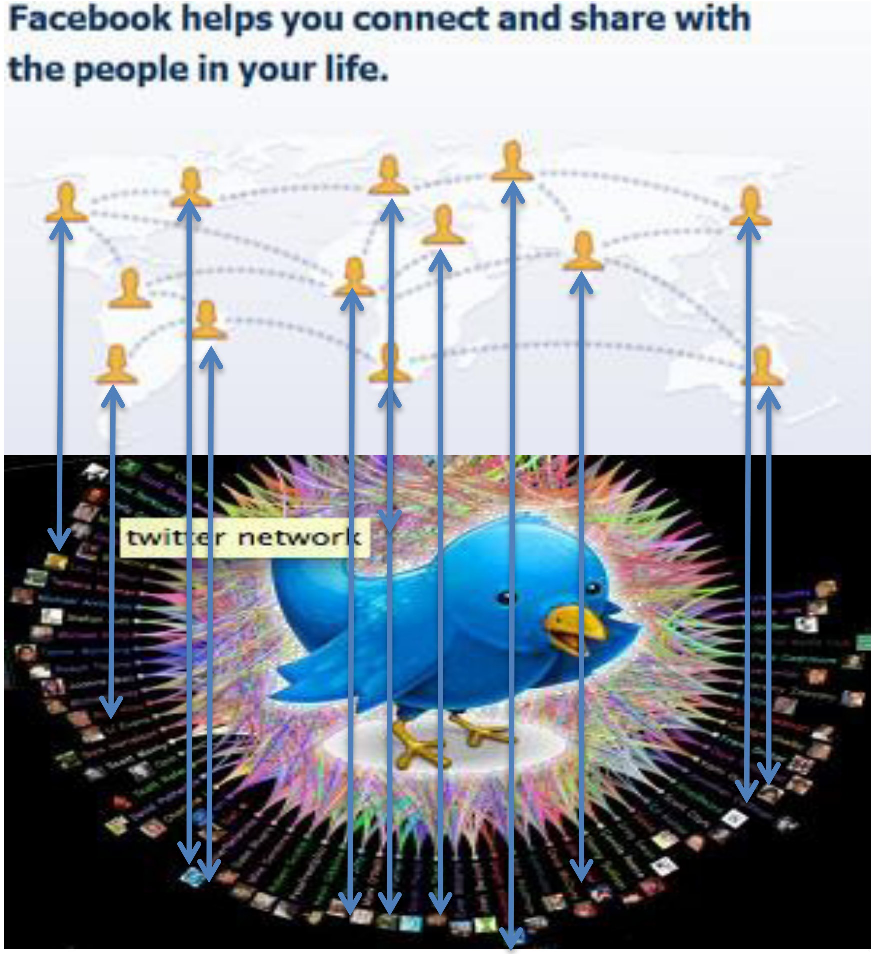
- Definitions
  - *Simple network* (or *graph*)
  - *Multigraph*: multiple links between nodes, equivalent to a weighted network with integer weights
  - *Hypergraph*: links may connect any number of nodes
  - *Simple network with colored edges*: links of different classes
  - *Multiplex*: network in layers, and with connections between layers; the interconnections between layers are only between a node and its counterpart in the other layer (i.e. the same node)

Instances of multiplex

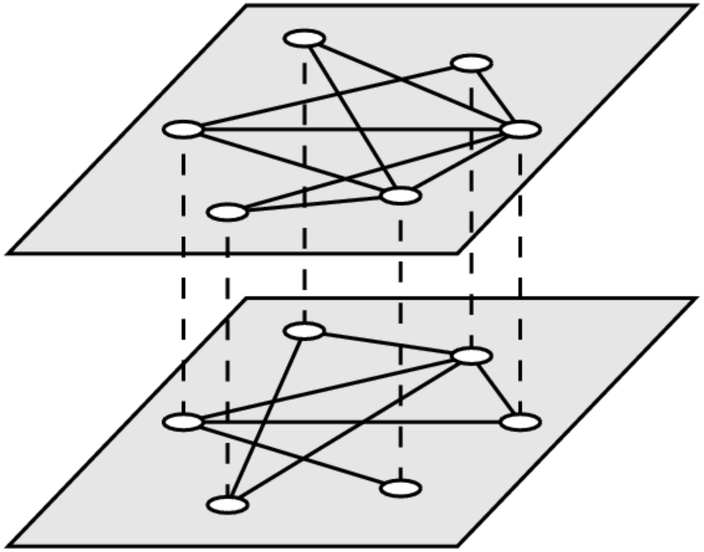




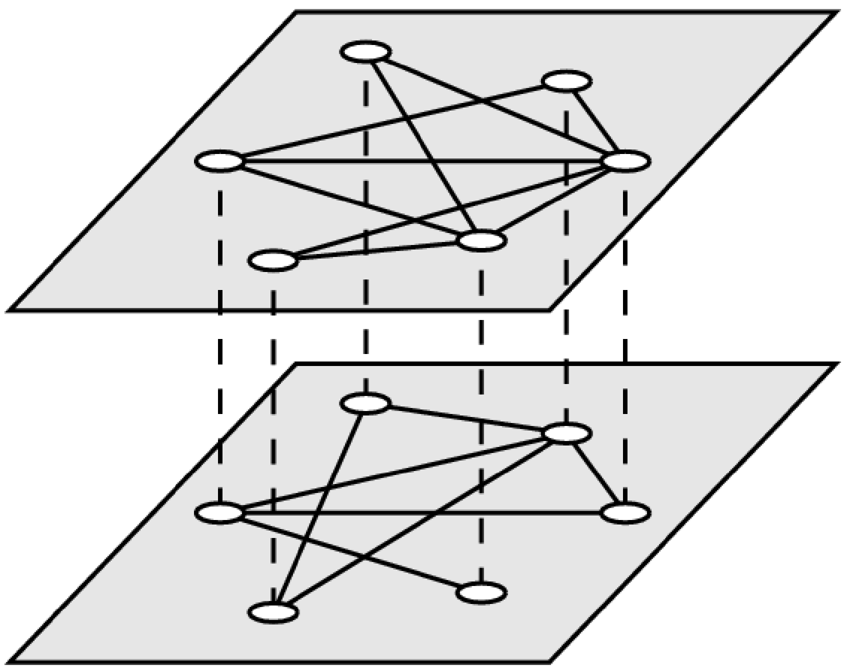




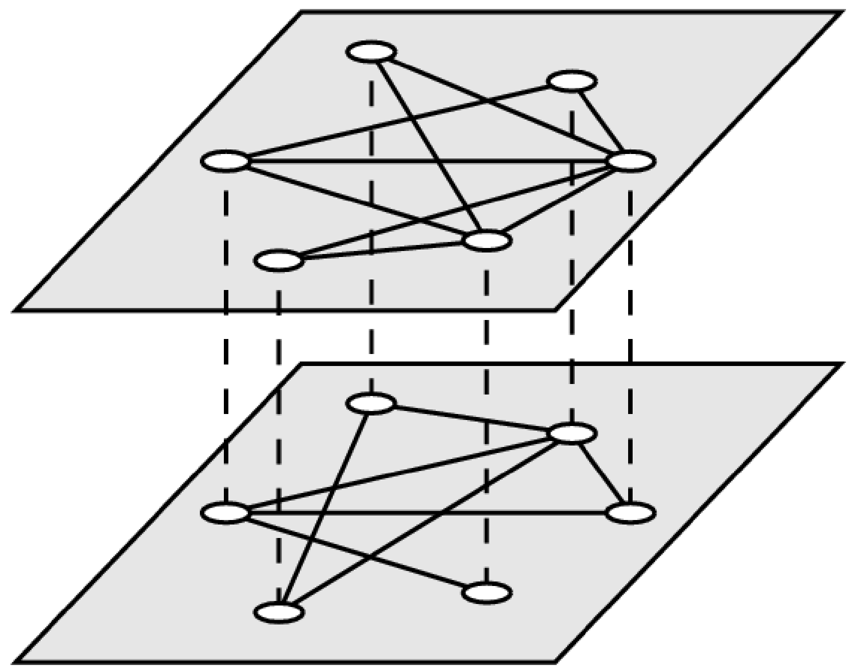
- Interest of multiplex networks:
  - Represent a “next step” in the study of the complexity of networked systems
  - Non trivial correlations emerge between different layers
  - The preceding studies on percolation in interdependent networks show that coupling networks can have unexpected emergent behaviors
  - Understood dynamical processes can show new characteristics because of the multiplex structure
  - The theory of multiplex must cover the current theory developed for “simplex” networks



- Dynamic processes on multiplex networks
  - Physics: Diffusion processes
  - Social sciences: Prisoner dilemma
  - Biology: Boolean dynamics



- **Diffusion in a multiplex network**
  - Objective
    - Which is the diffusion time scale in the multiplex?
      - Diffusion on multiplex networks, S. Gomez, A. Diaz-Guilera, J. Gomez-Gardenes, C.J. Perez-Vicente, Y. Moreno and A. Arenas, Physical Review Letters, 110, 028701 (2013)
  - Hypothesis
    - Only two layers
    - Connected weighted networks (W_1 and W_2) at each layer
    - Diffusion rates: D_1, D_2, D_x



- Mathematical description
  - Connectivity of the multiplex

$$\mathcal{W} = \left(\begin{array}{c|c} D_1 W_1 & 0 \\ \hline 0 & D_2 W_2 \end{array}\right) + D_x \left(\begin{array}{c|c} 0 & I \\ \hline I & 0 \end{array}\right)$$

  - Laplacians at each layer

$$L_K = S_K - W_K \qquad (S_K)_{ii} = s_i^K = \sum_j w_{ij}^K$$

  - Supra-Laplacian of the multiplex

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{D} = \left(\begin{array}{c|c} D_1 L_1 & 0 \\ \hline 0 & D_2 L_2 \end{array}\right) + D_x \left(\begin{array}{c|c} I & -I \\ \hline -I & I \end{array}\right)$$

  - Diffusion equation

$$\dot{\mathbf{x}} = -\mathcal{L}\mathbf{x}$$

  - Simplification (without loss of generalization)

$$D_1 = D_2 = 1$$

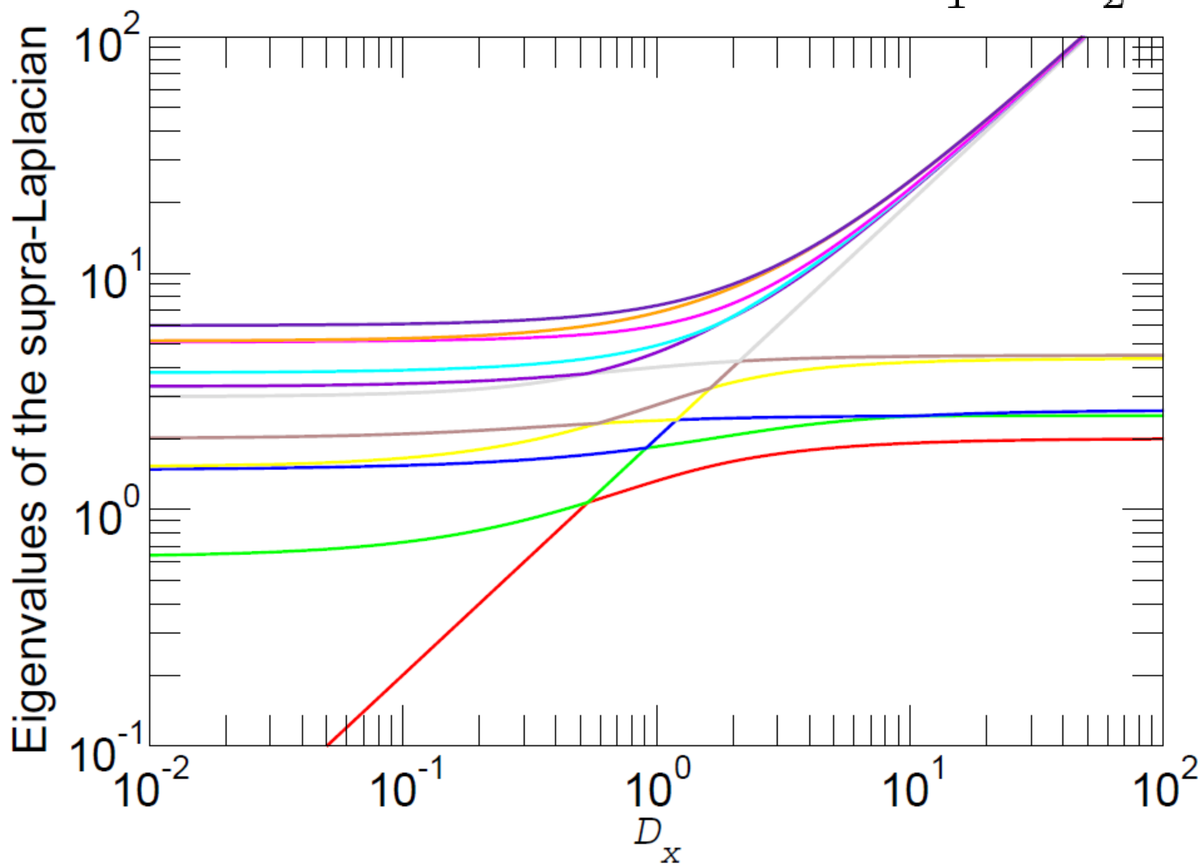
- Eigenvalues

$$0 = \lambda_1 < \lambda_2 \le \lambda_3 \le \ldots \le \lambda_{2n} \qquad D_x \neq 0$$

- Diffusion time scale

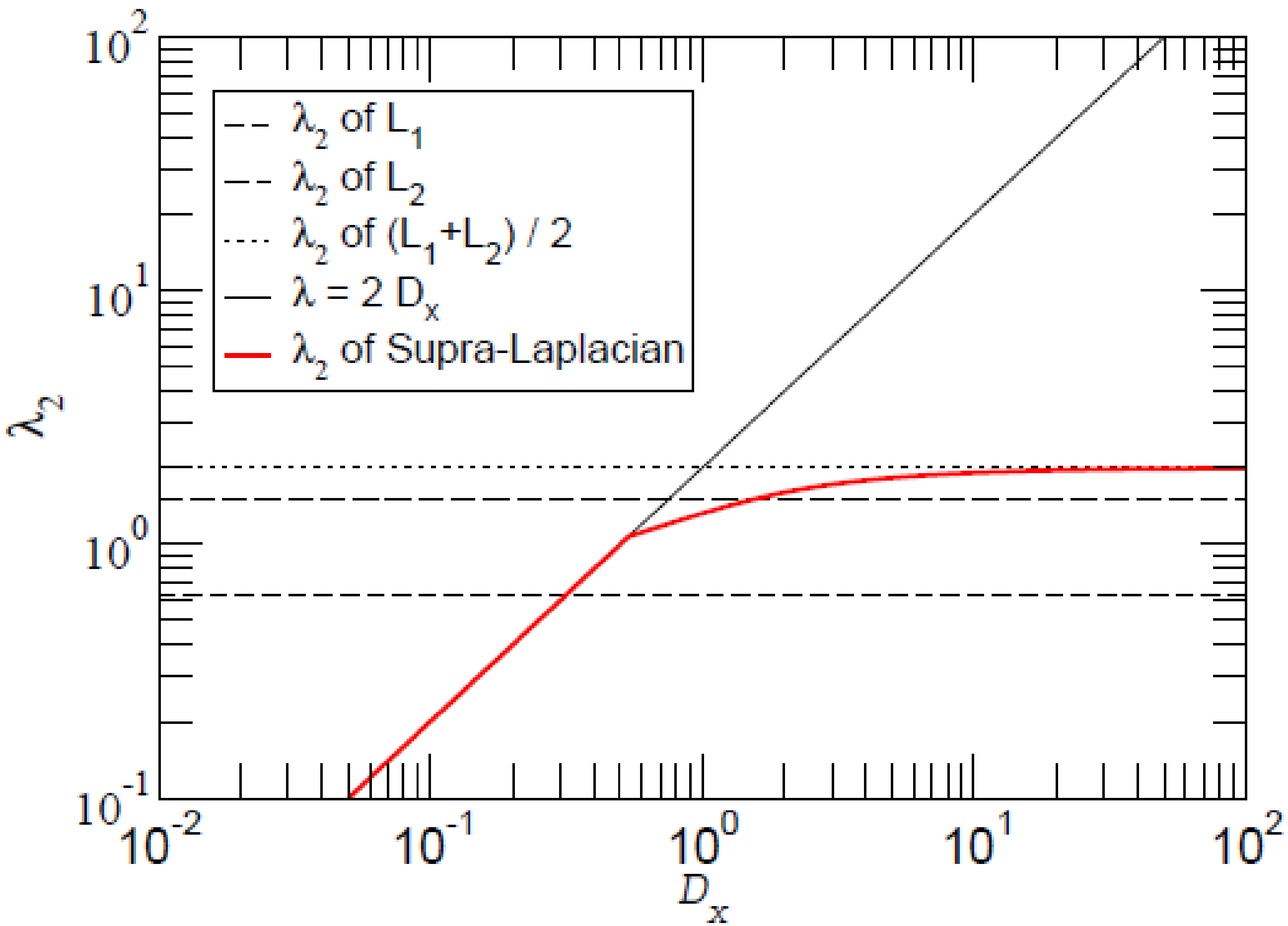
$$\tau = 1/\lambda_2$$

$$0 = \lambda_1^1 < \lambda_2^1 \le \ldots \lambda_n^1$$
$$0 = \lambda_1^2 < \lambda_2^2 \le \ldots \lambda_n^2$$



- Eigenvalues of supra-Laplacian for $D_x << 1$

$$L_1 \rightarrow \lambda_1^1 = 0 \rightarrow (1\cdots1|0\cdots0) \qquad (1\cdots1|1\cdots1) \rightarrow \lambda_1 = 0$$
$$L_2 \rightarrow \lambda_1^2 = 0 \rightarrow (0\cdots0|1\cdots1) \Rightarrow (1\cdots1|-1\cdots-1) \rightarrow \lambda = 2D_x$$



- Eigenvalues of supra-Laplacian for $D_x >> 1$

$$D_x = 1/\epsilon$$

$$\mathcal{L} = D_x \left[\left(\begin{array}{c|c} I & -I \\ \hline -I & I \end{array} \right) + \epsilon \left(\begin{array}{c|c} L_1 & 0 \\ \hline 0 & L_2 \end{array} \right) \right] = D_x \tilde{\mathcal{L}}$$

$$\left(\begin{array}{c|c} I & -I \\ \hline -I & I \end{array} \right) \Rightarrow \begin{array}{l} (\mathbf{u}|\mathbf{u}) \quad \rightarrow \tilde{\lambda} = 0 \\ (\mathbf{u}|-\mathbf{u}) \rightarrow \tilde{\lambda} = 2 \end{array}$$

At first order $O(\epsilon)$

$$\mathcal{L} \rightarrow \mathbf{v} = \left(\frac{\mathbf{u} + \epsilon \mathbf{u}'}{\mathbf{u} - \epsilon \mathbf{u}'} \right) \rightarrow \frac{\lambda_s}{2}$$

$$(L_1 + L_2)\mathbf{u} = \lambda_s \mathbf{u}$$

$$\mathbf{u}' = \frac{1}{4}(L_2 - L_1)\mathbf{u}$$

- Results
  - For $D_x << 1$

$$\tau = \frac{1}{2D_x}$$

  - For $D_x >> 1$

$$\tau \sim \frac{2}{\lambda_s}$$

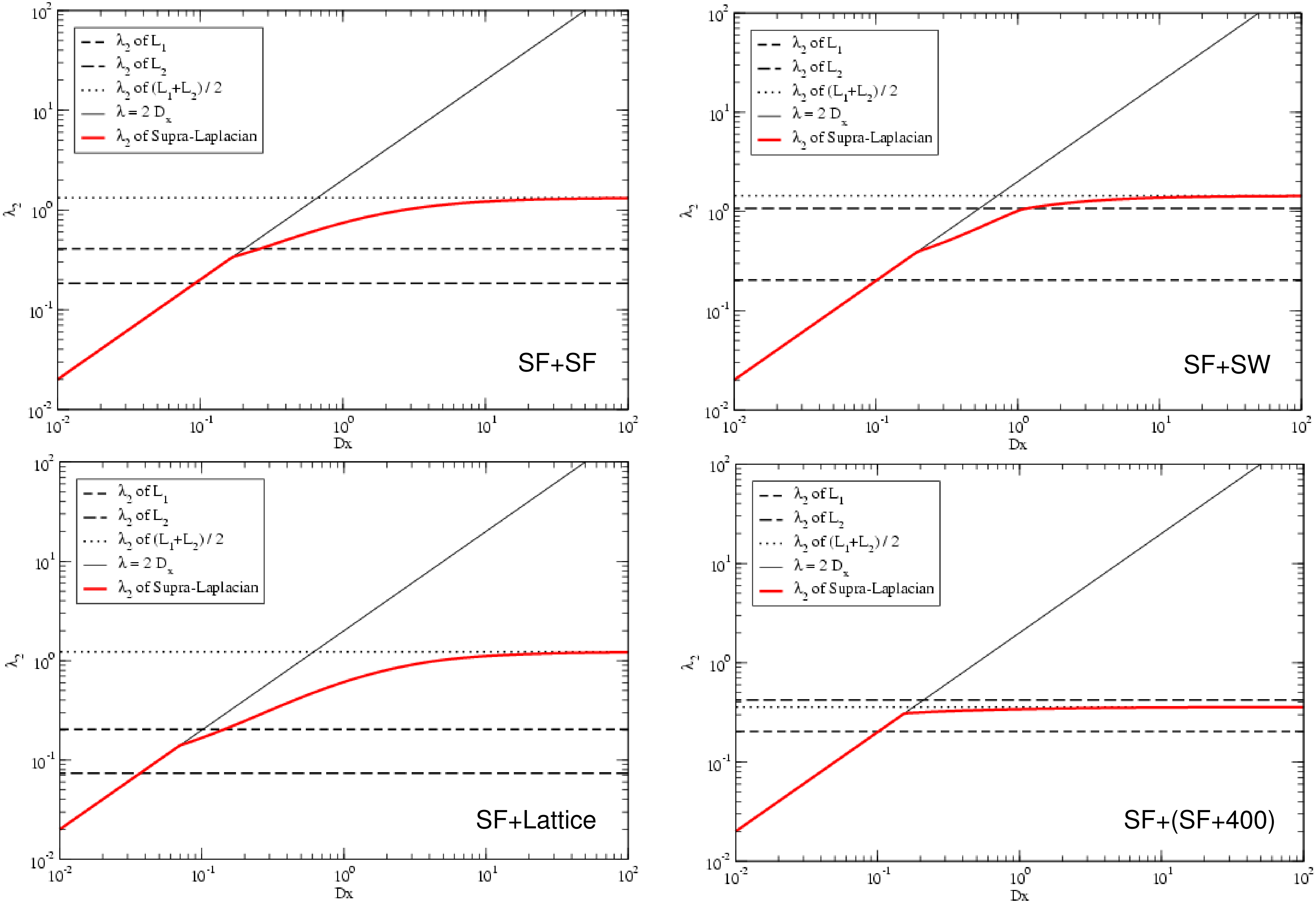
Multiplex faster than the slowest layer

$$\frac{\lambda_s}{2} \geq \frac{\lambda_2^1 + \lambda_2^2}{2} \geq \min\left(\lambda_2^1, \lambda_2^2\right)$$

Superdiffusion if multiplex faster than fastest layer

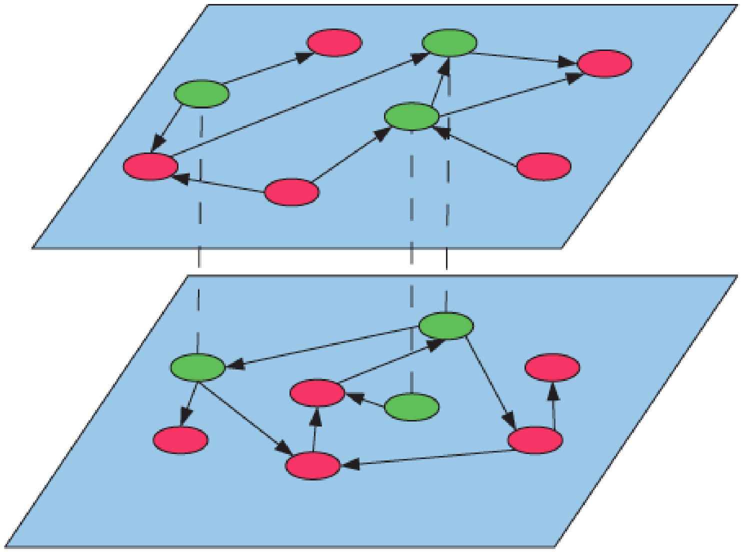
$$\frac{\lambda_s}{2} \leq \frac{2N}{2N-1} \min_i \left(\frac{s_i^1 + s_i^2}{2} + D_x \right)$$

Multiplex networks



- **Social dilemma on multiplex networks**
  - Objective
    - Understand prisoner’s dilemma on multiplex?
      - Evolution of Cooperation in Multiplex Networks, J. Gómez-Gardeñes, I. Reinares, A. Arenas and L. M. Floria, Scientific Reports, 10.1038 srep00620 (2012)
  - Hypothesis
    - Each node plays a PD

	C	D
C	1, 1	0, b
D	b, 0	0, 0

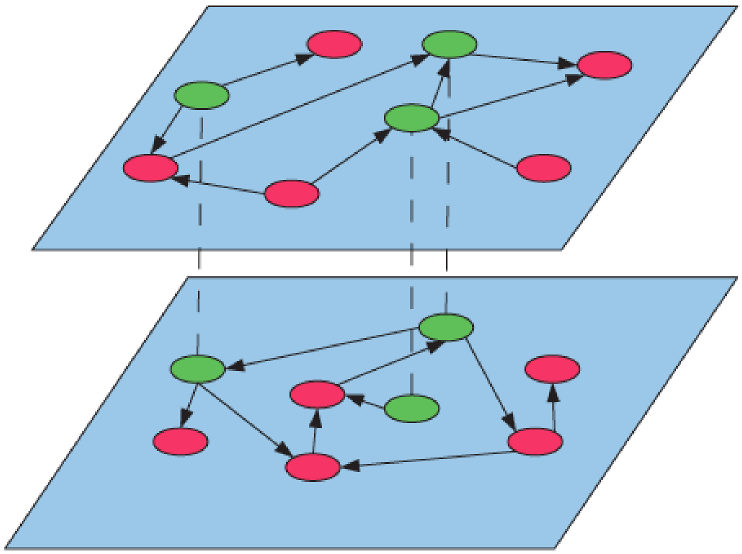


- Payoffs are accumulated through layers, update type Replicator-rule

- How it works:
- Each of the players, say *i*, chooses a layer, say *l*, at random among the *m* possible networks and a neighbor *j* (also randomly) among its k^l_i acquaintances. Then it compares their total payoffs, $P_i(t)$ and $P_j(t)$, obtained in the last round of the game.
- If $P_i(t) > P_j(t)$ nothing happens and *i* will use the same strategy within the network layer *l* in the next round of the PD game, $s_i(t + 1) = s_i(t)$.
- However, when $P_i(t) < P_j(t)$ agent *i* will take the strategy of *j* at layer *l*

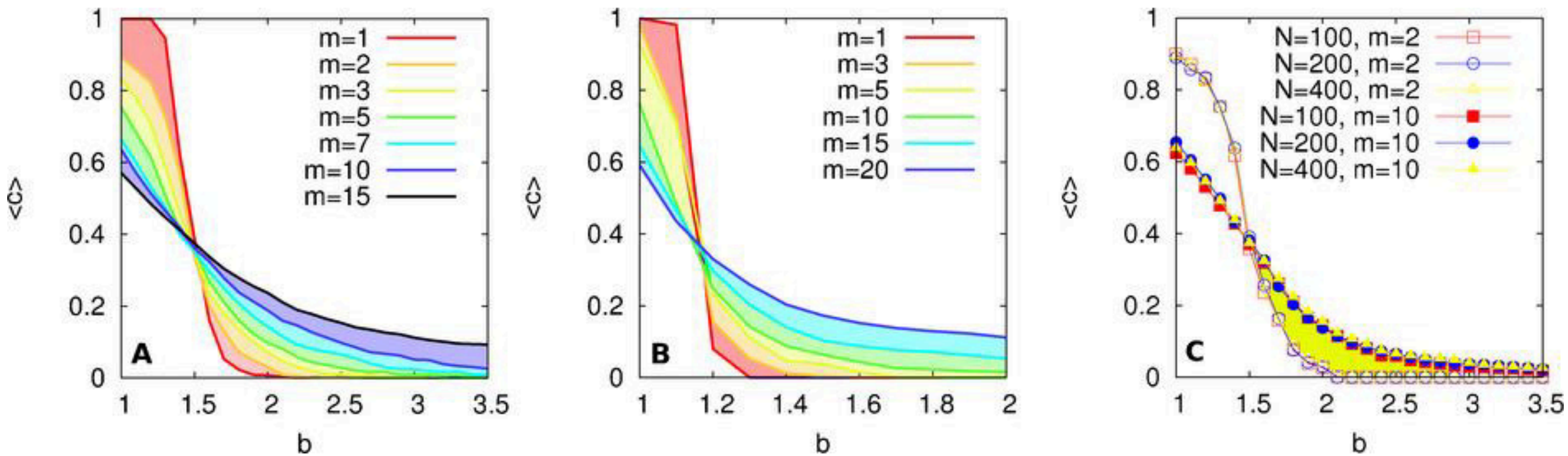
with a probability proportional to their payoff difference:

$$\Pi^l_{i \to j} = \frac{P_j(t) - P_i(t)}{b\, max(\sum_l k^l_i, \sum_l k^l_j)}$$



- Results:

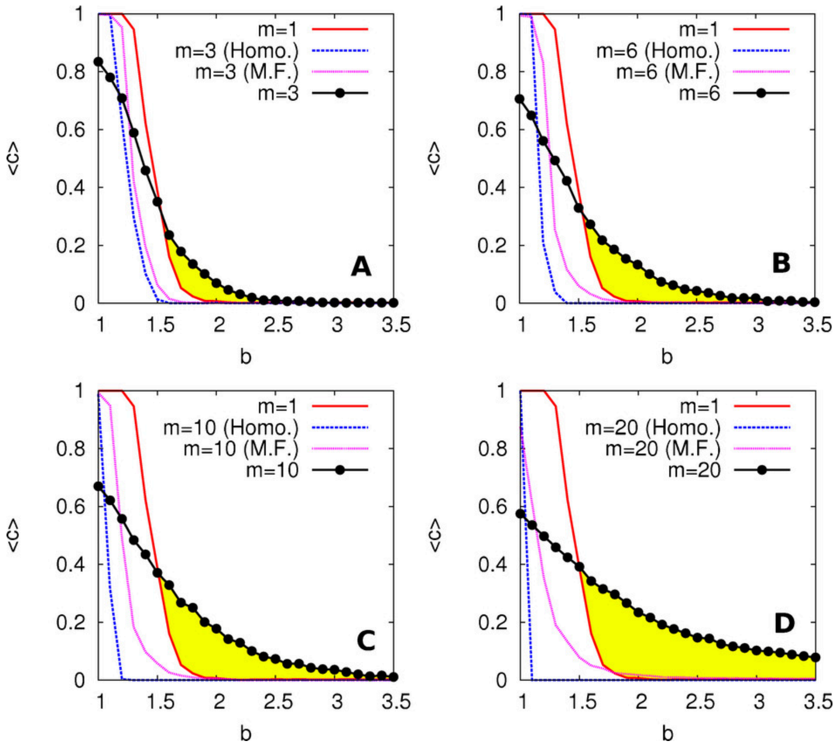
Average level of cooperation <*c*> as a function of the temptation to defect *b* for several multiplex with different number of layers *m*.



- In panel **A** the network layers are ER graphs with <*k*> = 3 (sparse graphs) while in panel **B** we have <*k*> = 20. In both cases *N* = 250 nodes. As can be observed, the resilience of cooperation increases remarkably as the number of layers *m* grows. Finally, panel **C** shows the curves <*c*>(*b*) for ER graphs with *k* = 3 (as in panel **A**) for *m* = 2 and *m* = 10 and different network sizes *N* = 100, 200 and 400.

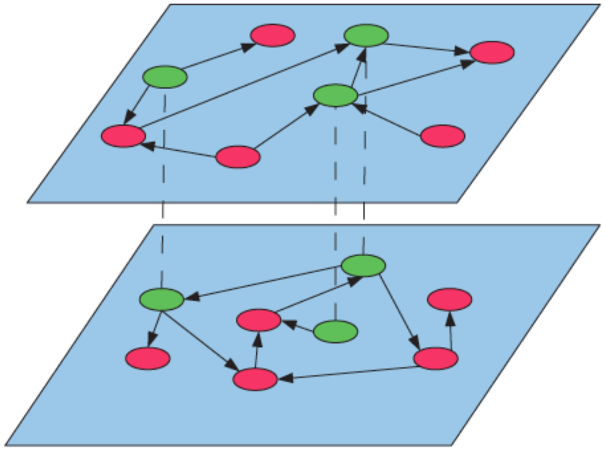
- Results

Average degree of cooperation *c* as a function of *b* (solid line with filled circles) for *m* = 3 (**A**), 6 (**B**), 10 (**C**) and 20 (**D**).



In each panel we show the case of a simplex (*m* = 1) network, the evolution <*c*>(*b*) for homogeneous strategists' populations corresponding to each value of *m* (Homo.) and the curve <*c*>(*b*) corresponding to the mean-field assumption for the coupling between layers (M.F.). Each multiplex network has *N* = 250 nodes while the interdependent layers are ER graphs with <*k*> = 3.

- **Stability of boolean multiplex networks**
  - Objective
    - Understand the interplay between structure and boolean dynamics?
      - Stability of Boolean multilevel networks, E. Cozzo, A. Arenas and Y. Moreno, PRE 86, 036115 (2012)
    - In cellular biochemical networks, many different signaling channels do work in parallel i.e., the same gene or biochemical specie can be involved in a regulatory interaction, in a metabolic reaction or in another signaling pathway.
  - Hypothesis
    - Boolean nodes (on-off)
    - Canalizing rules boolean functions
    - Observation of one layer with hidden connectivity of the degree in other layers



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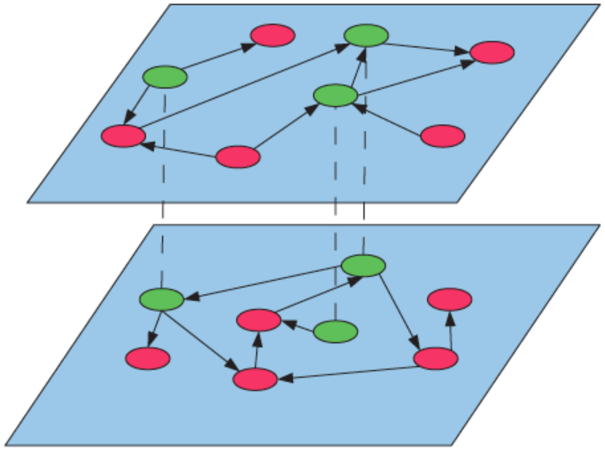
  - Hypothesis
    - Boolean nodes (on-off), not all nodes in all layers, Ñ(t) number of different nodes

$$\tilde{\mathbf{x}}(t) = (\tilde{x}_1(t), ..., \tilde{x}_{\tilde{N}(t)})$$

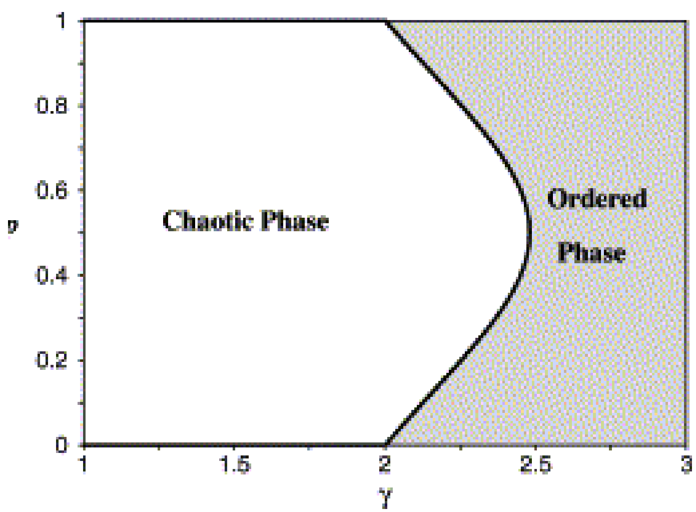


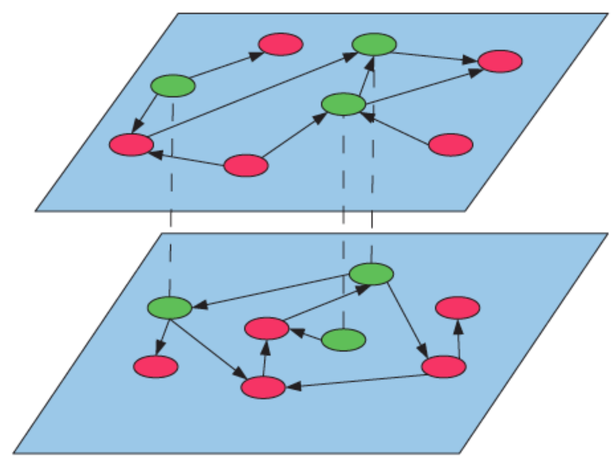
- Hypothesis
  - Update functions

$$\tilde{x}_i(t) = \tilde{f}_i(\tilde{x}_{j\in\Gamma^{\text{in}}_{\alpha}(i)}(t-1))$$

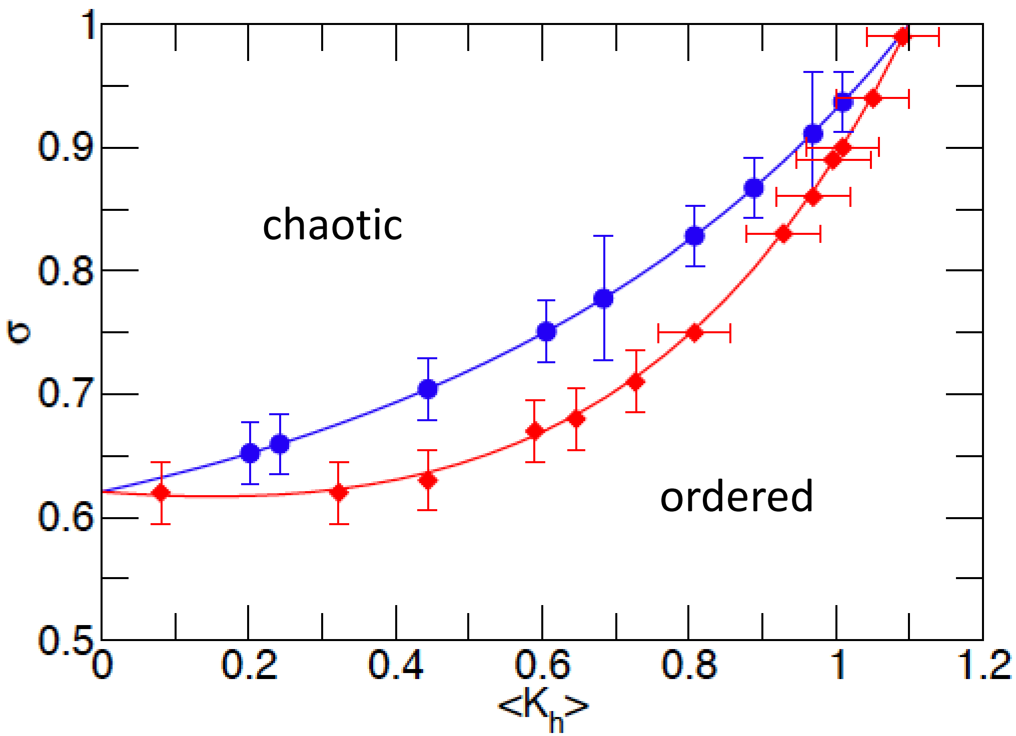
What is the effect of the multiplex connectivity?

Typical phase diagram of boolean networks





- Stabilizing effect of having several layers



Blue (mutiplex), red (simplex.)
Remarkably, the results show that a single ingredient − the multilevel nature of the system could explain why there are biologically stable systems that are however theoretically expected to operate in the unstable regime.

Summary:

- *Diffusion processes* on multiplex networks can show a non-trivial super-diffusion effect

- *The prisoner's dilemma* on top of multiplex networks coupled by the cumulative payoff prove to be more robust to cooperation

- *Boolean networks* in multiplex structures are able to stabilize the whole system while separate layers will show unstable regimes

- Papers and software available at
  - http://deim.urv.cat/~aarenas
  - http://deim.urv.cat/~alephsys